Implicit differentiation [47 marks]

1. [Maximum mark: 7] EXN.2.AHL.TZ0.6

The curve C has equation $\mathrm{e}^{2y} = x^3 + y$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2\mathrm{e}^{2y}-1}$. [3]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

attempts implicit differentiation on both sides of the equation **M1**

$2\mathrm{e}^{2y}\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + \frac{\mathrm{d}y}{\mathrm{d}x}$ **A1**

$\left(2\mathrm{e}^{2y} - 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$ **A1**

so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{2\mathrm{e}^{2y}-1}$ **AG**

[3 marks]

(b) The tangent to C at the point P is parallel to the y-axis.

Find the x-coordinate of P. [4]

Markscheme

attempts to solve $2\mathrm{e}^{2y} - 1 = 0$ for y **(M1)**

$y = -0.346\ldots \left(= \frac{1}{2}\ln\frac{1}{2}\right)$ **A1**

attempts to solve $\mathrm{e}^{2y} = x^3 + y$ for x given their value of y **(M1)**

$x = 0.946 \left(= \left(\frac{1}{2}\left(1 - \ln \frac{1}{2}\right)\right)^{\frac{1}{3}}\right)$ **A1**

[4 marks]

2. [Maximum mark: 8] 21N.2.AHL.TZ0.8

Consider the curve C given by $y = x - xy\ln(xy)$ where $x > 0,\ y > 0$.

(a) Show that $\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$. [3]

Markscheme

METHOD 1

attempts to differentiate implicitly including at least one application of the product rule *(M1)*

$$u = xy,\ v = \ln(xy),\ \frac{du}{dx} = x\frac{dy}{dx} + y,\ \frac{dv}{dx} = \left(x\frac{dy}{dx} + y\right)\frac{1}{xy}$$

$$\frac{dy}{dx} = 1 - \left[\frac{xy}{xy}\left(x\frac{dy}{dx} + y\right) + \left(x\frac{dy}{dx} + y\right)\ln(xy)\right]$$ ***A1***

Note: Award ***(M1)A1*** for implicitly differentiating $y = x(1 - y\ln(xy))$ and obtaining $\frac{dy}{dx} = 1 - \left[\frac{xy}{xy}\left(x\frac{dy}{dx} + y\right) + x\frac{dy}{dx}\ln(xy) + y\ln(xy)\right]$.

$$\frac{dy}{dx} = 1 - \left[\left(x\frac{dy}{dx} + y\right) + \left(x\frac{dy}{dx} + y\right)\ln(xy)\right]$$

$$\frac{dy}{dx} = 1 - \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy))$$ ***A1***

$$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$$ ***AG***

METHOD 2

$$y = x - xy \ln x - xy \ln y$$

attempts to differentiate implicitly including at least one application of the product rule *(M1)*

$$\frac{dy}{dx} = 1 - \left(\frac{xy}{x} + \left(x\frac{dy}{dx} + y\right)\ln x\right) - \left(\frac{xy}{y}\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)\ln y\right)$$

A1

or equivalent to the above, for example

$$\frac{dy}{dx} = 1 - \left(x \ln x\frac{dy}{dx} + (1 + \ln x)y\right) - \left(y \ln y + x\left(\ln y\frac{dy}{dx} + \frac{dy}{dx}\right)\right)$$

$$\frac{dy}{dx} = 1 - x\frac{dy}{dx}(\ln x + \ln y + 1) - y(\ln x + \ln y + 1)$$ ***A1***

or equivalent to the above, for example

$$\frac{dy}{dx} = 1 - x\frac{dy}{dx}(\ln (xy) + 1) - y(\ln (xy) + 1)$$

$$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$$ ***AG***

METHOD 3

attempt to differentiate implicitly including at least one application of the product rule ***M1***

$$u = x \ln (xy),\ v = y,\ \frac{du}{dx} = \ln(xy) + \left(x\frac{dy}{dx} + y\right)\frac{x}{xy},\ \frac{dv}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = 1 - \left(x\frac{dy}{dx}\ln(xy) + y \ln(xy) + \frac{xy}{xy}\left(x\frac{dy}{dx} + y\right)\right)$$ ***A1***

$$\frac{dy}{dx} = 1 - x\frac{dy}{dx}(\ln(xy) + 1) - y(\ln(xy) + 1)$$ ***A1***

$$\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right)(1 + \ln(xy)) = 1$$ ***AG***

METHOD 4

lets $w = xy$ and attempts to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ where $y = x - w \ln w$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}w}{\mathrm{d}x}\ln w\right) \left(= 1 - \frac{\mathrm{d}w}{\mathrm{d}x}(1 + \ln w)\right)$ ***A1***

$\frac{\mathrm{d}w}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\ln(xy)\right) \left(= 1 - \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(xy))\right)$

$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(xy)) = 1$ ***AG***

[3 marks]

(b) Hence find the equation of the tangent to C at the point where $x = 1$. [5]

Markscheme

METHOD 1

substitutes $x = 1$ into $y = x - xy\ln(xy)$ ***(M1)***

$y = 1 - y\ln y \Rightarrow y = 1$ ***A1***

substitutes $x = 1$ and their non-zero value of y into $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(xy)) = 1$ ***(M1)***

$2\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \left(\frac{\mathrm{d}y}{\mathrm{d}x} = 0\right)$ ***A1***

equation of the tangent is $y = 1$ ***A1***

METHOD 2

substitutes $x = 1$ into $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(xy)) = 1$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(y)) = 1$

EITHER

correctly substitutes $\ln y = \frac{1-y}{y}$ into $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(xy)) = 1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}\left(1 + \frac{1}{y}\right) = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 0\ (y = 1)$ ***A1***

OR

correctly substitutes $y + y \ln y = 1$ into
$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(xy)) = 1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}(2 + \ln y) = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 0\ (y = 1)$ ***A1***

THEN

substitutes $x = 1$ into $y = x - xy\ln(xy)$ ***(M1)***

$y = 1 - y \ln y \Rightarrow y = 1$

equation of the tangent is $y = 1$ ***A1***

[5 marks]

3. [Maximum mark: 15] 20N.1.AHL.TZ0.H_11

Consider the curve C defined by $y^2 = \sin(xy),\ y \neq 0$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\cos(xy)}{2y - x\cos(xy)}$. [5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt at implicit differentiation ***M1***

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(xy)\left[x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right]$$ ***A1M1A1***

Note: Award ***A1*** for LHS, ***M1*** for attempt at chain rule, ***A1*** for RHS.

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x}\cos(xy) + y\cos(xy)$$

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x}\cos(xy) = y\cos(xy)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(2y - x\cos(xy)) = y\cos(xy)$$ ***M1***

Note: Award ***M1*** for collecting derivatives and factorising.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\cos(xy)}{2y - x\cos(xy)}$$ ***AG***

[5 marks]

(b) Prove that, when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $y = \pm 1$. [5]

Markscheme

setting $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$y\cos(xy) = 0$ ***(M1)***

$(y \neq 0) \Rightarrow \cos(xy) = 0$ ***A1***

$\Rightarrow \sin\,(xy)\left(= \pm\sqrt{1-\cos^2\,(xy)} = \pm\sqrt{1-0}\right) = \pm 1$ ***OR***
$xy = (2n+1)\frac{\pi}{2}\,(n \in \mathbb{Z})$ ***OR*** $xy = \frac{\pi}{2},\ \frac{3\pi}{2}, \ldots$ ***A1***

Note: If they offer values for xy, award ***A1*** for at least two correct values in two different 'quadrants' and no incorrect values.

$y^2\,(= \sin\,(xy)) > 0$ ***R1***

$\Rightarrow y^2 = 1$ ***A1***

$\Rightarrow y = \pm 1$ ***AG***

[5 marks]

(c) Hence find the coordinates of all points on C, for $0 < x < 4\pi$, where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. [5]

Markscheme

$y = \pm 1 \Rightarrow 1 = \sin\,(\pm x) \Rightarrow \sin\,x = \pm 1$ OR
$y = \pm 1 \Rightarrow 0 = \cos\,(\pm x) \Rightarrow \cos\,x = 0$ ***(M1)***

$(\sin\,x = 1 \Rightarrow)\left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{2}, 1\right)$ ***A1A1***

$(\sin\,x = -1 \Rightarrow)\left(\frac{3\pi}{2}, -1\right), \left(\frac{7\pi}{2}, -1\right)$ ***A1A1***

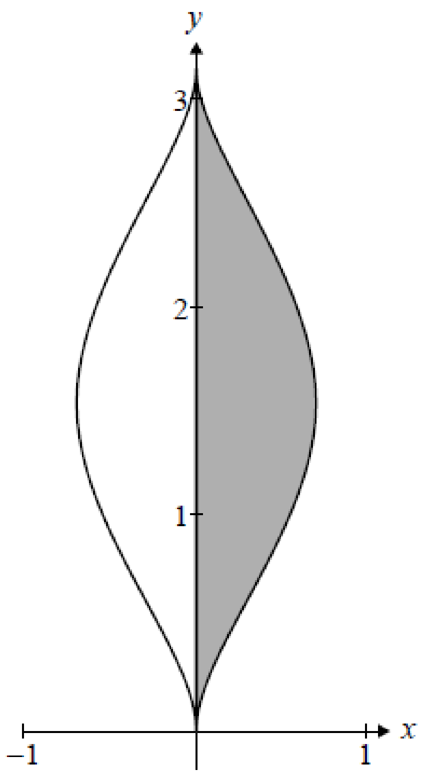
Note: Allow 'coordinates' expressed as $x = \frac{\pi}{2},\ y = 1$ for example.
Note: Each of the ***A*** marks may be awarded independently and are not dependent on ***(M1)*** being awarded.

Note: Mark only the candidate's first two attempts for each case of $\sin\,x$.

[5 marks]

4. [Maximum mark: 8] 19N.2.AHL.TZ0.H_11

The following diagram shows part of the graph of $2x^2 = \sin^3 y$ for $0 \leqslant y \leqslant \pi$.



(a.i) Using implicit differentiation, find an expression for $\frac{dy}{dx}$. [4]

Markscheme

valid attempt to differentiate implicitly ***(M1)***

$4x = 3\sin^2 y \cos y \frac{dy}{dx}$ ***A1A1***

$\frac{dy}{dx} = \frac{4x}{3\sin^2 y \cos y}$ ***A1***

[4 marks]

(a.ii) Find the equation of the tangent to the curve at the point $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$. [4]

Markscheme

at $\left(\frac{1}{4}, \frac{5\pi}{6}\right)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x}{3\sin^2 y \cos y} = \frac{1}{3\left(\frac{1}{2}\right)^2\left(-\frac{\sqrt{3}}{2}\right)}$ ***(M1)***

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{8}{3\sqrt{3}} (= -1.54)$ ***A1***

hence equation of tangent is

$y - \frac{5\pi}{6} = -1.54\left(x - \frac{1}{4}\right)$ **OR** $y = -1.54x + 3.00$ ***(M1)A1***

Note: Accept $y = -1.54x + 3$.

[4 marks]

5. [Maximum mark: 9] 19M.1.AHL.TZ1.H_7

Find the coordinates of the points on the curve $y^3 + 3xy^2 - x^3 = 27$ at which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. [9]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt at implicit differentiation ***M1***

$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2 + 6xy\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^2 = 0$ ***A1A1***

Note: Award ***A1*** for the second & third terms, ***A1*** for the first term, fourth term & RHS equal to zero.

substitution of $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1***

$3y^2 - 3x^2 = 0$

$\Rightarrow y = \pm x$ ***A1***

substitute either variable into original equation ***M1***

$y = x \Rightarrow x^3 = 9 \Rightarrow x = \sqrt[3]{9}$ (or $y^3 = 9 \Rightarrow y = \sqrt[3]{9}$) ***A1***

$y=-x \Rightarrow x^3=27 \Rightarrow x=3$ (or $y^3=-27 \Rightarrow y=-3$) ***A1***

$\left(\sqrt[3]{9},\ \sqrt[3]{9}\right)$, (3, −3) ***A1***

[9 marks]