D MODELLING PERIODIC BEHAVIOUR

The sine and cosine functions are both referred to as **sinusoidal functions**. They can be used to model many periodic phenomena in the real world. In some cases, such as the movement of the hands on a clock, the models we find will be almost exact. In other cases, such as the maximum daily temperature of a city over a year, the model will be less accurate.

Example 5 — Self Tutor

The average daytime temperature for a city is given by the function $D(t) = 5\cos\left(\frac{\pi}{6}t\right) + 20$ °C, where t is the time in months after January.

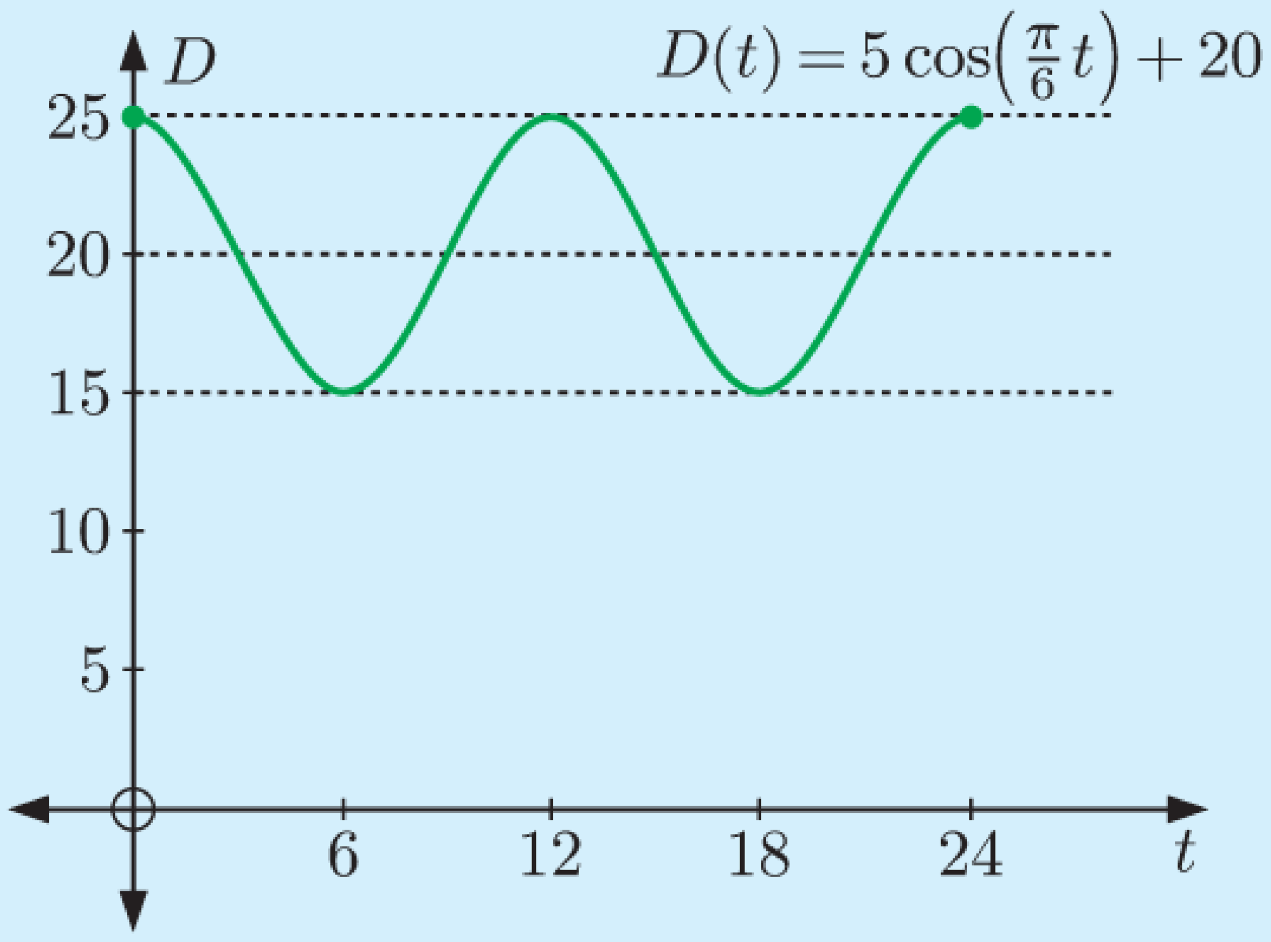
a Sketch the graph of D against t for $0 \leqslant t \leqslant 24$.

b Find the average daytime temperature during May.

c Find the minimum average daytime temperature, and the month in which it occurs.

a For $D(t) = 5\cos\left(\frac{\pi}{6}t\right) + 20$:

- the amplitude is 5
- the period is $\frac{2\pi}{(\frac{\pi}{6})} = 12$ months
- the principal axis is $D = 20$.



b May is 4 months after January.

When $t = 4$, $D = 5 \times \cos\frac{4\pi}{6} + 20$

$= 5 \times (-\frac{1}{2}) + 20$

$= 17.5$

So, the average daytime temperature during May is 17.5°C.

c The minimum average daytime temperature is $20 - 5 = 15$°C, which occurs when $t = 6$ or 18.

So, the minimum average daytime temperature occurs during July.

EXERCISE 17D

1 The temperature inside Vanessa's house t hours after midday is given by the function $T(t) = 6\sin\left(\frac{\pi}{12}t\right) + 26$ °C.

a Sketch the graph of T against t for $0 \leqslant t \leqslant 24$.

b Find the temperature inside Vanessa's house at:

i midnight **ii** 2 pm.

c Find the maximum temperature inside Vanessa's house, and the time at which it occurs.

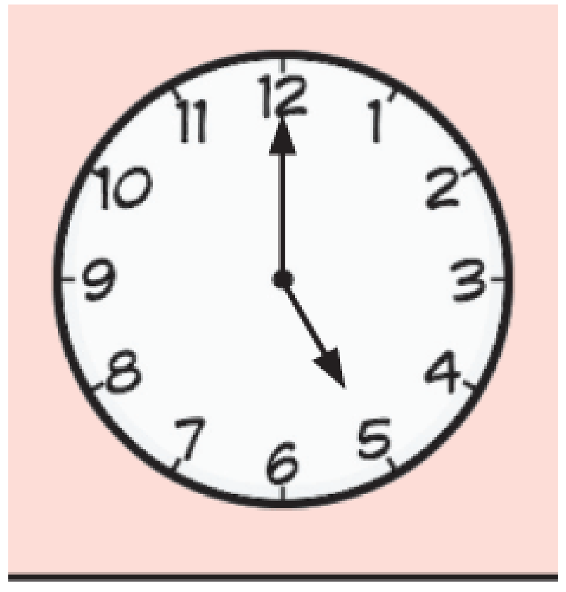
2 The depth of water in a harbour t hours after midnight is $D(t) = 4\cos\left(\frac{\pi}{6}t\right) + 6$ metres.

a Sketch the graph of D against t for $0 \leqslant t \leqslant 24$.

b Find the highest and lowest depths of the water, and the times at which they occur.

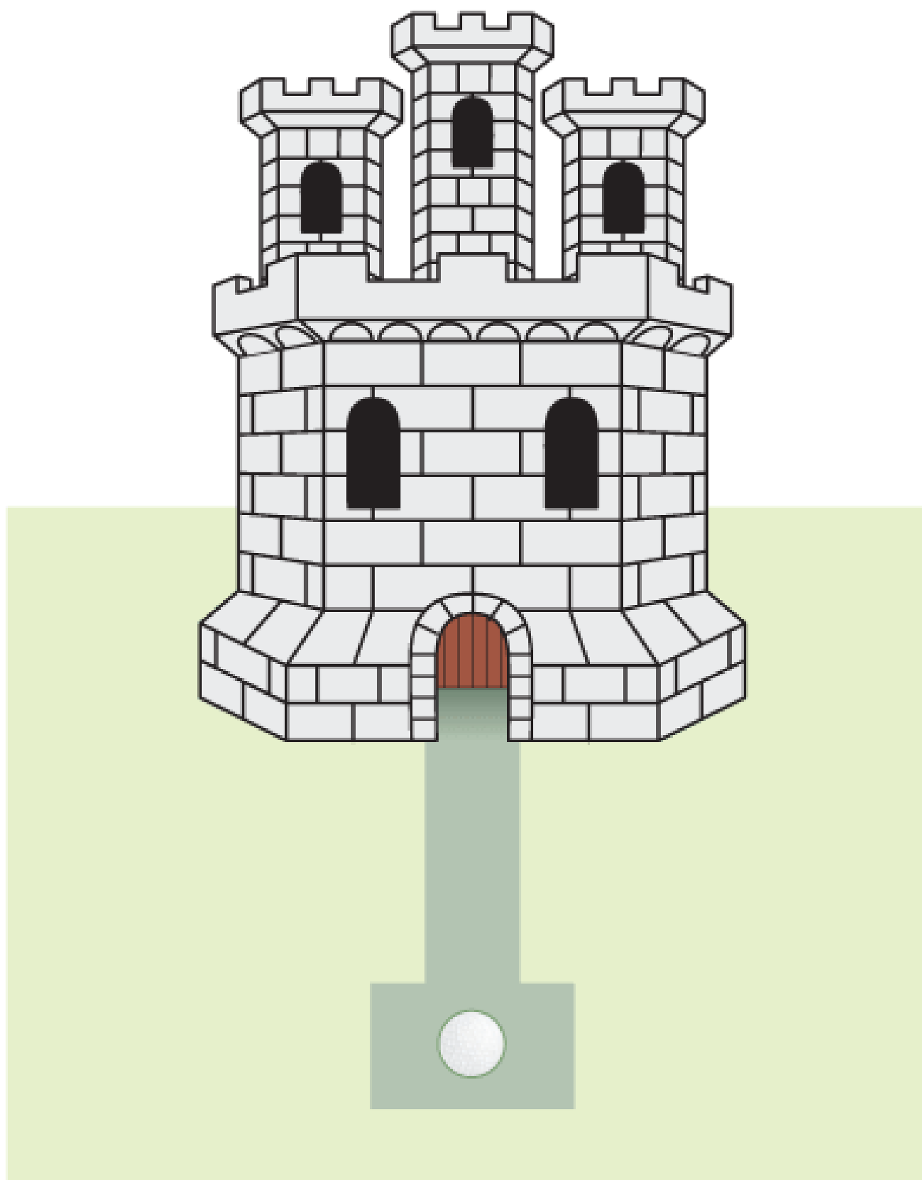
c A boat requires a water depth of 5 metres to sail in. Will the boat be able to enter the harbour at 8 pm?

3 The tip of a clock's minute hand is $H(t) = 15\cos\left(\frac{\pi}{30}t\right) + 150$ cm above ground level, where t is the time in minutes after 5 pm.



   a Sketch the graph of H against t for $0 \leqslant t \leqslant 180$.
   b Find the length of the minute hand.
   c Find, rounded to 1 decimal place, the height of the minute hand's tip at:
      i 5:08 pm **ii** 5:37 pm **iii** 5:51 pm **iv** 6:23 pm

4 On a mini-golf hole, golfers must putt the ball through a castle's entrance. The entrance is protected by a gate which moves up and down.
The height of the gate above the ground t seconds after it touches the ground is $H(t) = 4\sin\left(\frac{\pi}{4}(t-2)\right) + 4$ cm.



   a Sketch the graph of H against t for $0 \leqslant t \leqslant 16$.
   b Find the height of the gate above the ground 2 seconds after the gate touches the ground.
   c Eric is using a golf ball with radius 2.14 cm. He putts the ball 1 second after the gate touches the ground, and the ball takes 5.3 seconds to reach the castle's entrance. Will the ball pass through the entrance?

Example 6 **Self Tutor**

On a hot summer day in Madrid, Antonio pays careful attention to the temperature. The maximum of 41.8°C occurs at 3:30 pm. The minimum was 27.5°C. Suggest a sine function to model the temperature for that day.

The mean temperature $= \dfrac{41.8 + 27.5}{2} = 34.65$°C, so $d = 34.65$.

The amplitude $= \dfrac{41.8 - 27.5}{2} = 7.15$°C

$\therefore\ a = 7.15$

The period is 24 hours, so $b = \frac{2\pi}{24} = \frac{\pi}{12}$.

The maximum occurs at 3:30 pm, so we assume the temperature passed its mean value 6 hours earlier, at 9:30 am.

The day begins at midnight, so the function is shifted $9\frac{1}{2}$ hours to the right, thus $c = 9.5$.

If t is the number of hours after midnight, the temperature T is modelled by

$$T(t) = 7.15\sin\left(\tfrac{\pi}{12}(t - 9.5)\right) + 34.65\ ^{\circ}\text{C}.$$

5 On a September day in Moscow, the maximum temperature 15.8°C occurred at 2 pm. The minimum was 5.4°C. Suggest a sine function to model the temperature for that day. Let T be the temperature and t be the time in hours after midnight.

6 The ferry operator at Picton, New Zealand, is studying the tides. High tides occur every 12.4 hours. The first high tide tomorrow will be at 1:30 am. The high tide will be 1.36 m and the low tide will be 0.16 m. Find a cosine function to model the tide height for the day. Let H be the tide height and t be the time in hours after midnight.

7 Answer the **Opening Problem** on page **448**.

8 Some of the largest tides in the world are observed in Canada's Bay of Fundy. The difference between high and low tides is 14 metres, and the average time difference between high tides is about 12.4 hours. On a particular day, the first high tide is 16.2 m, occurring at 9 am.

a Find a sine model for the height of the tide H in terms of the time t.

b Sketch the graph of the function for that day.

9 On an analogue clock, the hour hand is 6 cm long and the minute hand is 12 cm long. Let t be the time in hours after midnight.

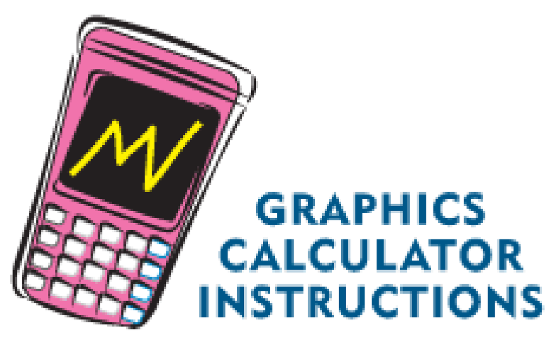
a Write a cosine function for the height of the tip of the hour hand relative to the centre of the clock.

b Write a sine function for the horizontal displacement of the tip of the minute hand relative to the centre of the clock.

E FITTING TRIGONOMETRIC MODELS TO DATA

Suppose we have **data** in which we observe periodic behavior. In such cases, we usually cannot fit an *exact* model. However, we can still apply the same principles to estimate the period, amplitude, and principal axis from the data.

You can check your models using your graphics calculator. Click on this icon for instructions.



Example 7 Self Tutor

The mean monthly maximum temperatures for Cape Town, South Africa are shown below:

Month (t)	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temperature (T °C)	28	27	25.5	22	18.5	16	15	16	18	21.5	24	26

We want to model the data with a trigonometric function of the form $T = a\sin(b(t-c)) + d$ where Jan $\equiv 1$, Feb $\equiv 2$, and so on.

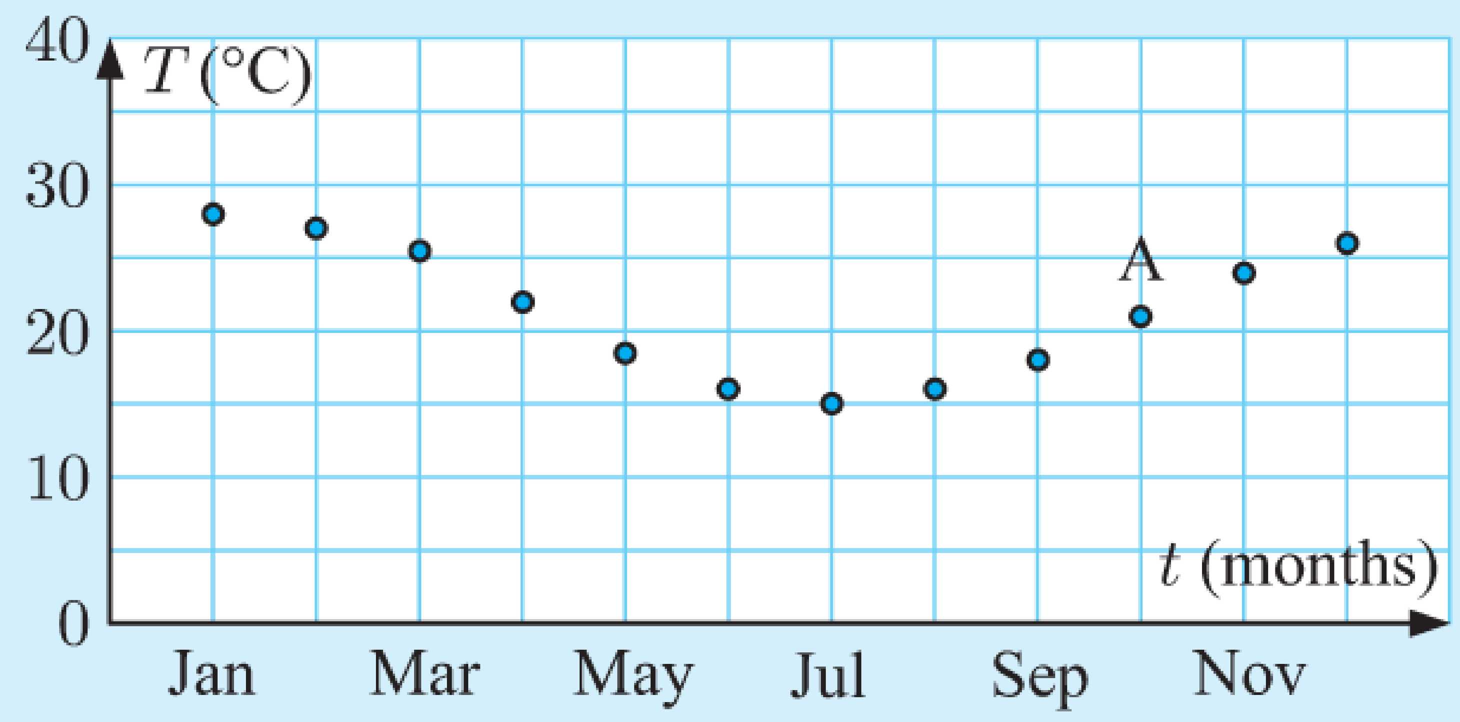
a Draw a scatter diagram of the data.

b Without using technology, estimate:

i b **ii** a **iii** d **iv** c

c Check your answers using technology.

a



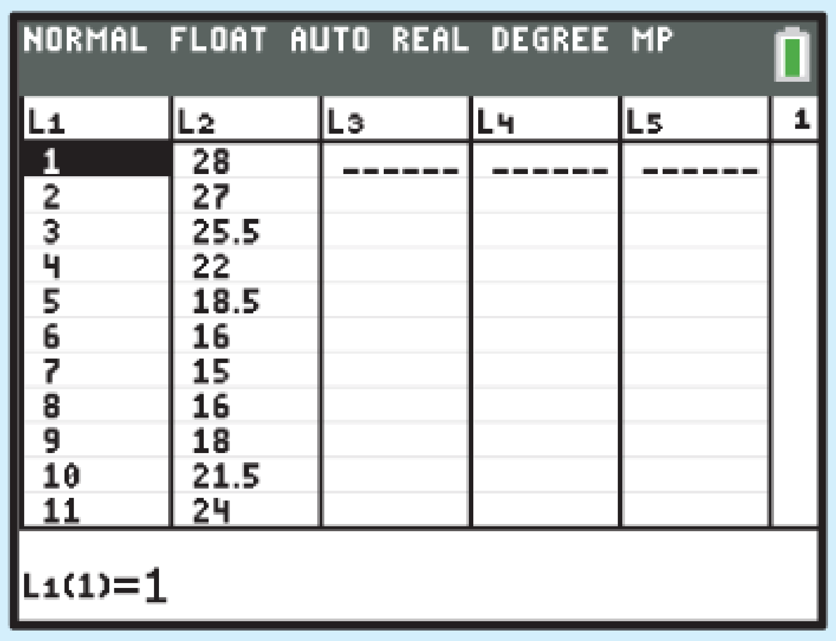
b **i** The period is 12 months, so $\frac{2\pi}{b} = 12$ and $\therefore$ $b = \frac{\pi}{6}$.

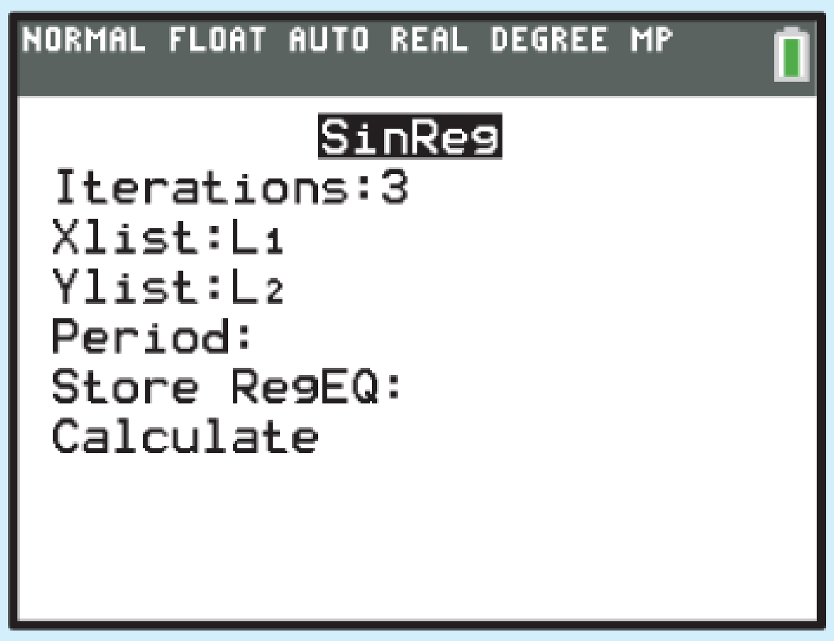
ii The amplitude $= \frac{\max - \min}{2} \approx \frac{28 - 15}{2} \approx 6.5$, so $a \approx 6.5$.

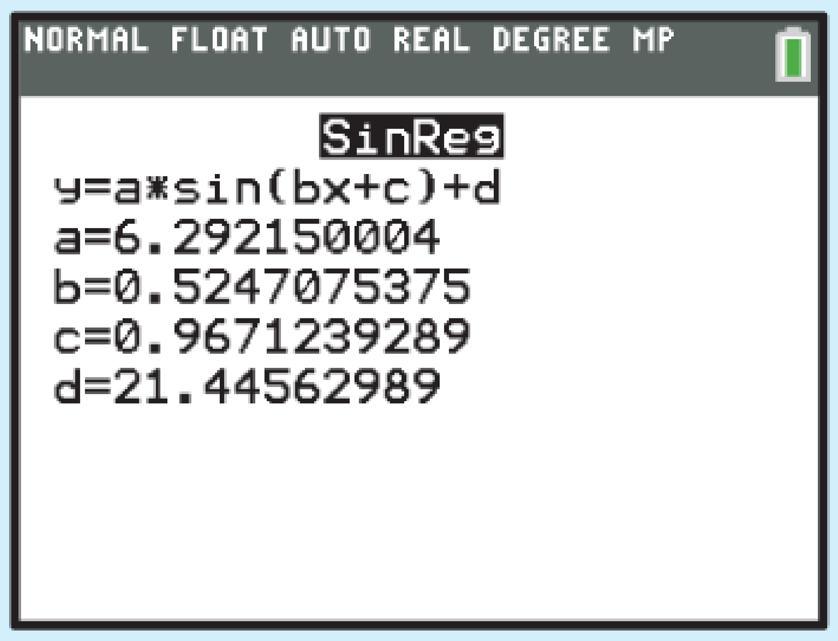
iii The principal axis is midway between the maximum and minimum, so $d \approx \frac{28+15}{2} \approx 21.5$.

iv The model is $T \approx 6.5\sin\left(\frac{\pi}{6}(t - c)\right) + 21.5$ for some constant c.
On the original graph, point A is the first point shown at which the sine function starts a new period. Since A is at $(10, 21.5)$, $c = 10$.

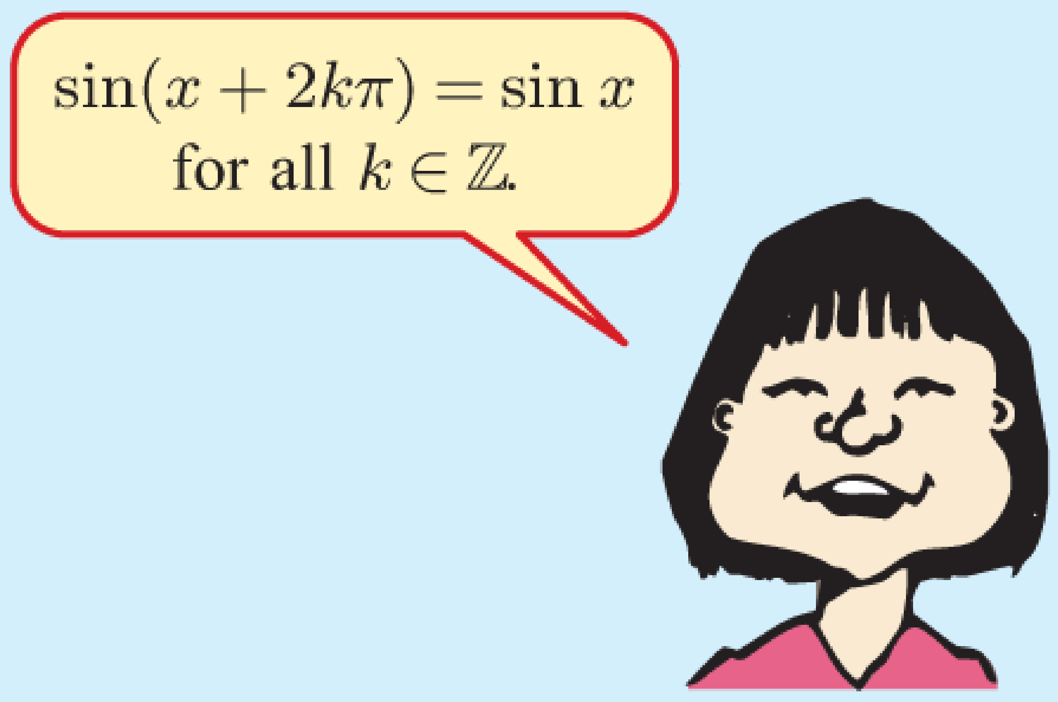
c From **b**, our model is $T \approx 6.5\sin\left(\frac{\pi}{6}(t-10)\right) + 21.5$
$\approx 6.5\sin(0.524t - 5.24) + 21.5$







Using technology,
$T \approx 6.29\sin(0.525t + 0.967) + 21.4$
$\approx 6.29\sin(0.525t + 0.967 - 2\pi) + 21.4$
$\approx 6.29\sin(0.525t - 5.32) + 21.4$



EXERCISE 17E

1 Below is a table which shows the mean monthly maximum temperatures for a city in Greece.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	14	15	18	21	25	27	26	24	20	18	16

a Draw a scatter diagram of the data.

b What features of the data suggest a trigonometric model is appropriate?

c Your task is to model the data with a sine function of the form $T \approx a\sin(b(t-c)) + d$, where Jan $\equiv 1$, Feb $\equiv 2$, and so on.
Without using technology, estimate:

i b **ii** a **iii** d **iv** c

d Use technology to check your model. How well does your model fit?

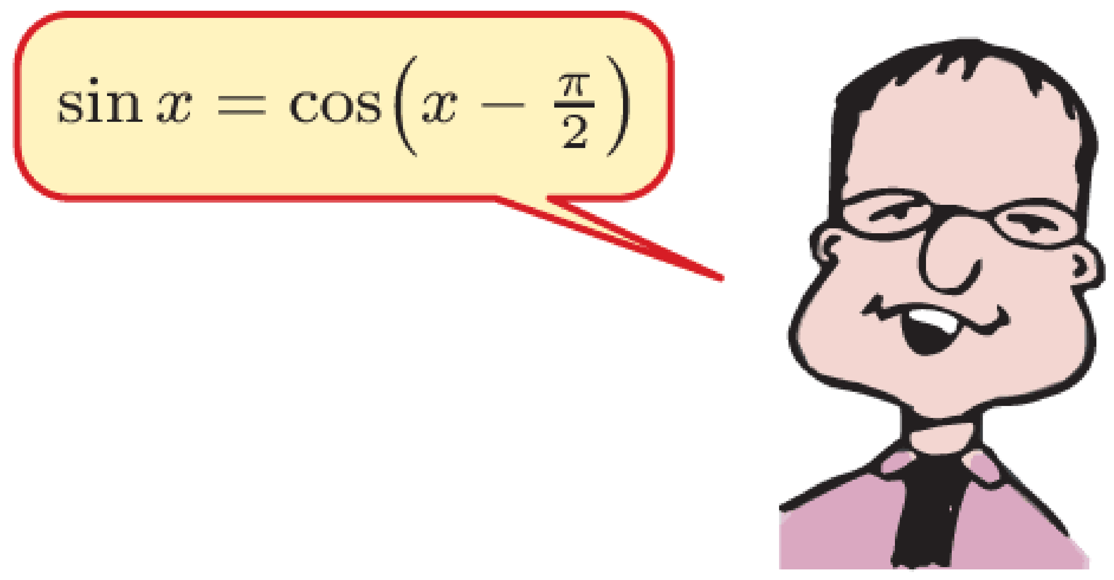
2 The data in the table shows the mean monthly temperatures for Christchurch, New Zealand.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14

a Find a cosine model for this data in the form $T \approx a\cos(b(t-c))+d$ without using technology. Let Jan $\equiv 1$, Feb $\equiv 2$, and so on.

b Draw a scatter diagram of the data and sketch the graph of your model on the same set of axes.

c Use technology to check your answer to **a**.



3 At the Mawson base in Antarctica, the mean monthly temperatures for the last 30 years are:

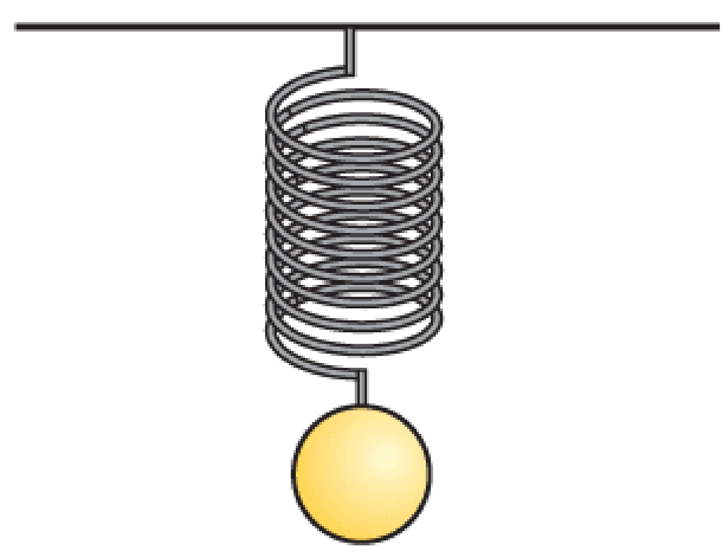
Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	0	0	−4	−9	−14	−17	−18	−19	−17	−13	−6	−2

a Find a sine model for this data without using technology. Use Jan $\equiv 1$, Feb $\equiv 2$, and so on.

b Draw a scatter diagram of the data and sketch the graph of your model on the same set of axes.

c How appropriate is the model?

4 An object is suspended from a spring. If the object is pulled below its resting position and then released, it will oscillate up and down. The data below shows the height of the object relative to its rest position, at different times.



Time (t seconds)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Height (H cm)	−15	−13	−7.5	0	7.5	13	15	13	7.5	0

Time (t seconds)	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Height (H cm)	−7.5	−13	−15	−13	−7.5	0	7.5	13	15	13	7.5

a Draw a scatter diagram of the data.

b Find a trigonometric function which models the height of the object over time.

c Use your model to predict the height of the object after 4.25 seconds.

d What do you think is unrealistic about this model? What would happen differently in reality?

RESEARCH

1 How accurately will a trigonometric function model the phases of the moon?

2 Are there any periodic phenomena which can be modelled by the *sum* of trigonometric functions?

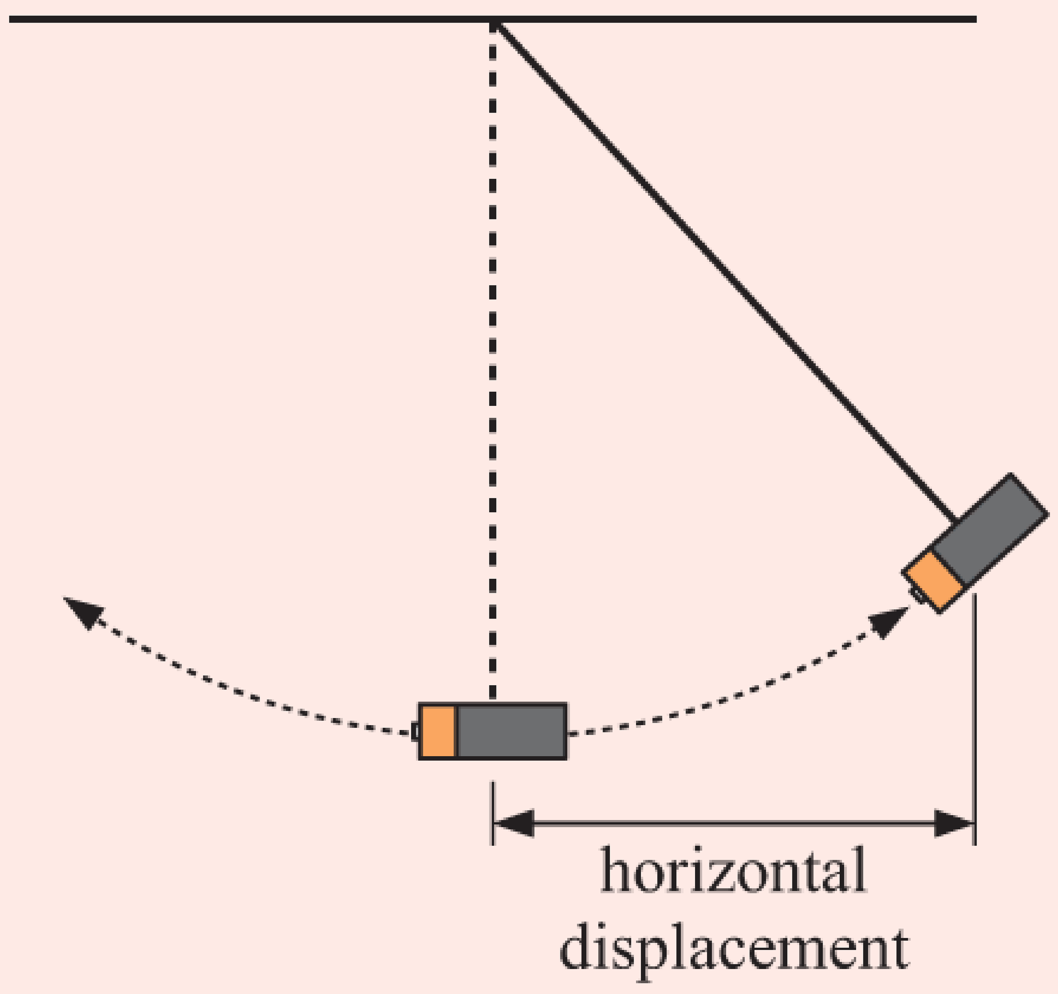
ACTIVITY 2 THE PENDULUM

In this Activity you will work in small groups to model the behaviour of a pendulum.

You will need: string, sticky tape, a ruler, a stopwatch, and a AA battery.

What to do:

1. Cut a piece of string of length 75 cm. Attach one end of the string to the battery, and the other end to your desk.
2. Hold the battery to one side, then release it, causing the battery to swing back and forth like a pendulum.
3. Using your stopwatch and ruler, measure the maximum and minimum horizontal displacement reached by the battery, and the times at which they occurred. You may need to repeat the experiment several times, but make sure the battery is released from the same position each time.



4. Use your data to find a trigonometric function which models the horizontal displacement of the battery over time.
5. What part of the function affects the *period* of the pendulum?
6. Repeat the experiment with strings of different length. Explore the relationship between the length of the string and the period of the pendulum.

F THE TANGENT FUNCTION

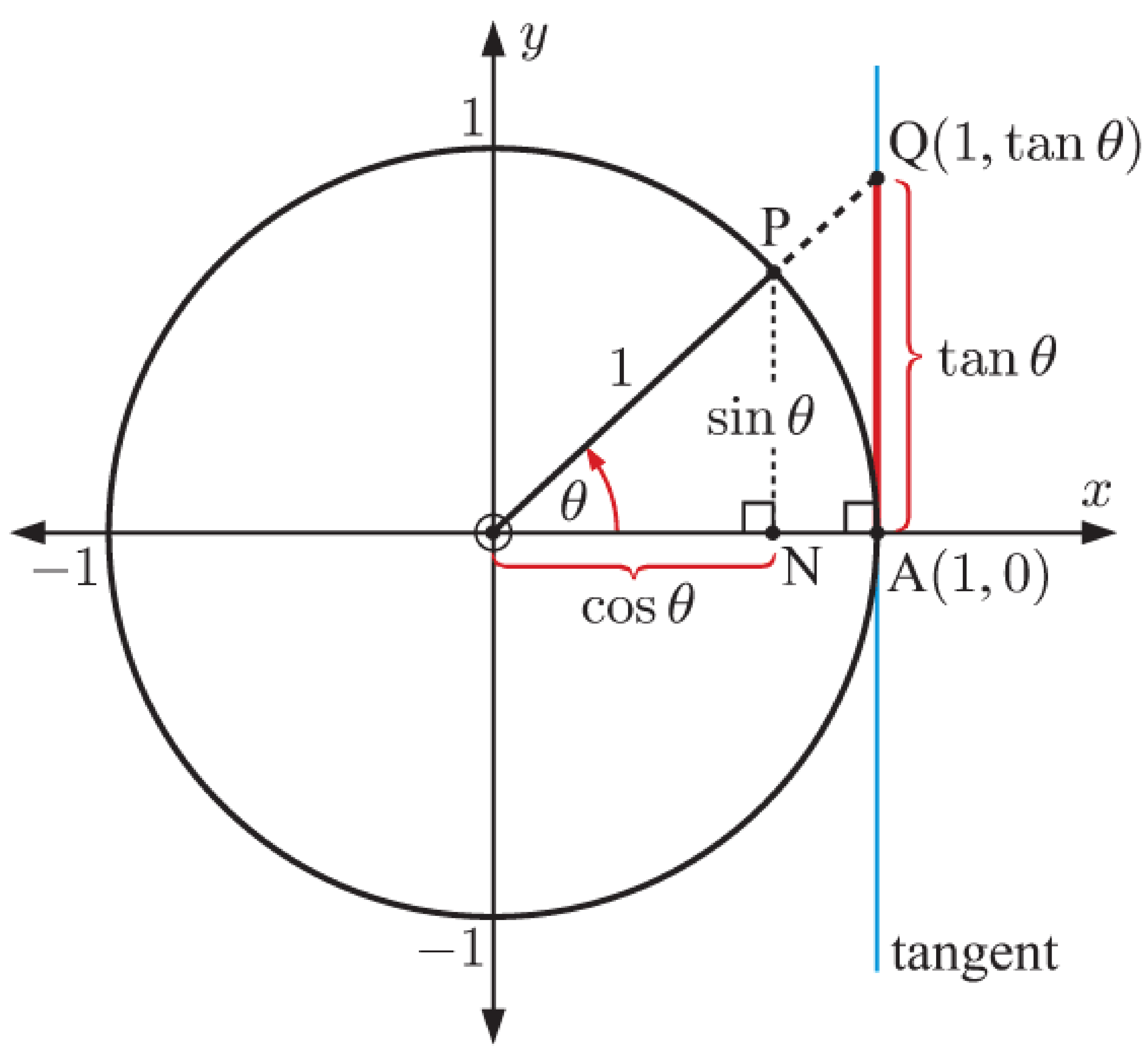
We have seen that if $P(\cos\theta, \sin\theta)$ is a point which is free to move around the unit circle, and if [OP] is extended to meet the tangent at $A(1, 0)$, the intersection between these lines occurs at $Q(1, \tan\theta)$.

This enables us to define the **tangent function**

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

We have also seen that $\tan\theta$ is:

- positive in quadrants 1 and 3
- negative in quadrants 2 and 4
- periodic with period π.



DISCUSSION

What happens to $\tan\theta$ when P is at:

a $(1, 0)$ and $(-1, 0)$

b $(0, 1)$ and $(0, -1)$?