Revision (31.01) [210 marks]

1. [Maximum mark: 20]

The function f is defined by $f(x) = \cos^2 x - 3\sin^2 x,\ \ 0 \leq x \leq \pi$.

(a) Find the roots of the equation $f(x) = 0$. [5]

Markscheme

$\cos^2 x - 3\sin^2 x = 0$

valid attempt to reduce equation to one involving one trigonometric function ***(M1)***

$\frac{\sin^2 x}{\cos^2 x} = \frac{1}{3}$ OR $1 - \sin^2 x - 3\sin^2 x = 0$ OR $\cos^2 x - 3(1 - \cos^2 x) = 0$ OR $\cos 2x - 1 + \cos 2x = 0$

correct equation ***(A1)***

$\tan^2 x = \frac{1}{3}$ OR $\cos^2 x = \frac{3}{4}$ OR $\sin^2 x = \frac{1}{4}$ OR $\cos 2x = \frac{1}{2}$

$\tan x = \pm\frac{1}{\sqrt{3}}$ OR $\cos x = \pm\frac{\sqrt{3}}{2}$ OR $\sin x = (\pm)\frac{1}{2}$ OR $2x = \frac{\pi}{3}\left(, \frac{5\pi}{3}\right)$ ***(A1)***

$x = \frac{\pi}{6},\ x = \frac{5\pi}{6}$ ***A1A1***

Note: Award ***M1A1A0A1A0*** for candidates who omit the $\pm$ (for tan or cos) and give only $x = \frac{\pi}{6}$.
Award ***M1A1A0A0A0*** for candidates who omit the $\pm$ (for tan or cos) and give only $x = 30^\circ$.

Award ***M1A1A1A1A0*** for candidates who give both answers in degrees.

Award ***M1A1A1A1A0*** for candidates who give both correct answers in radians, but who include additional solutions outside the domain.

Award a maximum of ***M1A0A0A1A1*** for correct answers with no working.

[5 marks]

(b.i) Find $f'(x)$.

[2]

Markscheme

attempt to use the chain rule (may be evidenced by at least one $\cos x \sin x$ term) *(M1)*

$f\prime(x) = -2 \cos x \sin x - 6 \sin x \cos x (= -8 \sin x \cos x = -4 \sin 2x)$ *A1*

[2 marks]

(b.ii) Hence find the coordinates of the points on the graph of $y = f(x)$ where $f\prime(x) = 0$. [5]

Markscheme

valid attempt to solve their $f\prime(x) = 0$ ***(M1)***

At least 2 correct x-coordinates (may be seen in coordinates) ***(A1)***

$x = 0,\ x = \frac{\pi}{2},\ x = \pi$

Note: Accept additional correct solutions outside the domain.
Award ***A0*** if any additional incorrect solutions are given.

correct coordinates (may be seen in graph for part (c)) ***A1A1A1***

$(0, 1),\ (\pi, 1),\ \left(\frac{\pi}{2}, -3\right)$

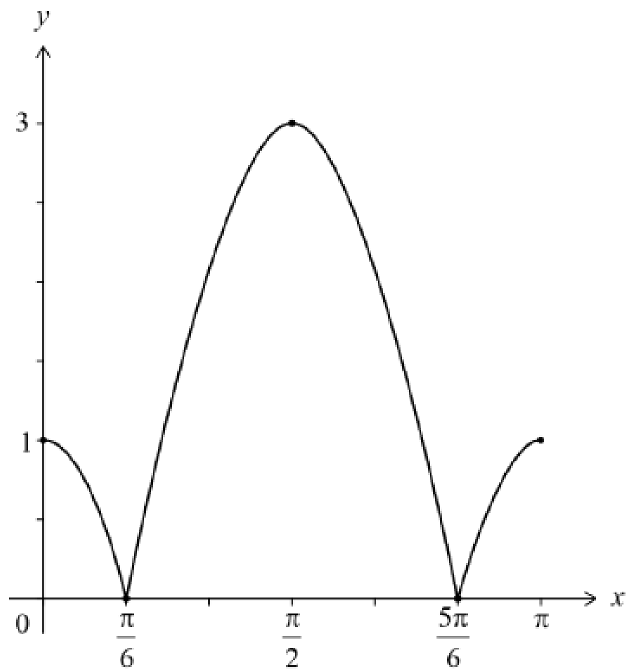
Note: Award a maximum of ***M1A1A1A1A0*** if any additional solutions are given.

Note: If candidates do not find at least two correct x-coordinates, it is possible to award the appropriate final marks for their correct coordinates, such as **M1A0A0A1A0**.

[5 marks]

(c) Sketch the graph of $y = |f(x)|$, clearly showing the coordinates of any points where $f\prime(x) = 0$ and any points where the graph meets the coordinate axes. [4]

Markscheme



attempt to reflect the negative part of the graph of f in the x-axis ***M1***

endpoints have coordinates $(0, 1),\ (\pi, 1)$ ***A1***

smooth maximum at $\left(\frac{\pi}{2}, 3\right)$ ***A1***

sharp points (cusps) at x-intercepts $\frac{\pi}{6},\ \frac{5\pi}{6}$ ***A1***

[4 marks]

(d) Hence or otherwise, solve the inequality $|f(x)| > 1$. [4]

Markscheme

considers points of intersection of $y = |f(x)|$ and $y = 1$ on graph or algebraically ***(M1)***

$-\left(\cos^2 x - 3\sin^2 x\right) = 1$ or $-\left(1 - 4\sin^2 x\right) = 1$ or
$-\left(4\cos^2 x - 3\right) = 1$ or $-(2\cos 2x - 1) = 1$

$\tan^2 x = 1$ or $\sin^2 x = \frac{1}{2}$ or $\cos^2 x = \frac{1}{2}$ or $\cos 2x = 0$ *(A1)*

$x = \frac{\pi}{4}, \frac{3\pi}{4}$ *(A1)*

For $|f(x)| > 1$

$\frac{\pi}{4} < x < \frac{3\pi}{4}$ *A1*

[4 marks]

2. [Maximum mark: 9]

Let $f(x) = \frac{x^2 - 10x + 5}{x+1}$, $x \in \mathbb{R}$, $x \neq -1$.

(a) Find the co-ordinates of all stationary points. [4]

Markscheme

$f'(x) = \frac{(2x-10)(x+1) - (x^2-10x+5)1}{(x+1)^2}$ *M1*

$f'(x) = 0 \Rightarrow x^2 + 2x - 15 = 0 \Rightarrow (x+5)(x-3) = 0$ *M1*

Stationary points are $(-5, \ -20)$ and $(3, \ -4)$ *A1A1*

[4 marks]

(b) Write down the equation of the vertical asymptote. [1]

Markscheme

$x = -1$ *A1*

[1 mark]

(c) With justification, state if each stationary point is a minimum, maximum or horizontal point of inflection. [4]

Markscheme

Looking at the nature table

x		-5		-1		3	
$f'(x)$	+ve	0	-ve	undefined	-ve	0	+ve

M1A1

$(-5,\ -20)$ is a max and $(3,\ -4)$ is a min ***A1A1***

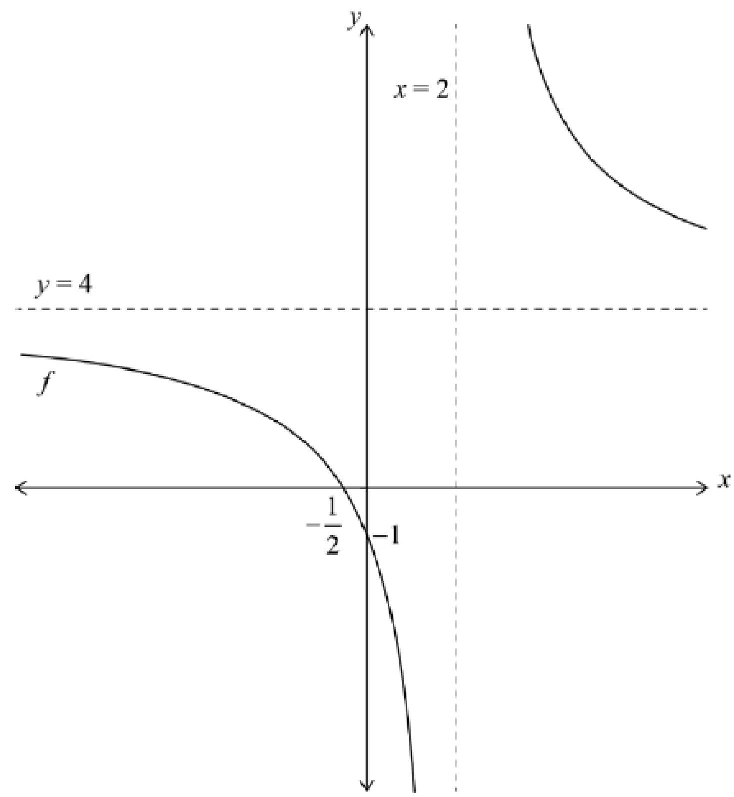
[4 marks]

3. [Maximum mark: 16]

Consider the function $f(x) = \frac{4x+2}{x-2},\ x \neq 2$.

(a) Sketch the graph of $y = f(x)$. On your sketch, indicate the values of any axis intercepts and label any asymptotes with their equations. [5]

Markscheme



vertical asymptote $x = 2$ sketched and labelled with correct equation ***A1***

horizontal asymptote $y = 4$ sketched and labelled with correct equation ***A1***

For an approximate rational function shape:

labelled intercepts $-\frac{1}{2}$ on x-axis, -1 on y-axis ***A1A1***

two branches in correct opposite quadrants with correct asymptotic behaviour ***A1***

Note: These marks may be awarded independently.

[5 marks]

(b) Write down the range of f. [1]

Markscheme

$y \neq 4$ (or equivalent) ***A1***

[1 mark]

Consider the function $g(x) = x^2 + bx + c$. The graph of g has an axis of symmetry at $x = 2$.

The two roots of $g(x) = 0$ are $-\frac{1}{2}$ and p, where $p \in \mathbb{Q}$.

(c) Show that $p = \frac{9}{2}$. [1]

Markscheme

$2 + \frac{5}{2}$ OR $\left(-\frac{1}{2}\right) + 2 \times \frac{5}{2}$ OR $\frac{-\frac{1}{2}+p}{2} = 2$ OR $-4 = -p + \frac{1}{2}$ ***A1***

$p = \frac{9}{2}$ ***AG***

[1 mark]

(d) Find the value of b and the value of c. [3]

Markscheme

METHOD 1

attempt to substitute both roots to form a quadratic ***(M1)***

EITHER

$\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right)$ OR $x^2 - \left(-\frac{1}{2} + \frac{9}{2}\right)x + \left(-\frac{1}{2} \times \frac{9}{2}\right)$

$= x^2 - 4x - \frac{9}{4}$ ***A1A1***

$\left(b = -4, c = -\frac{9}{4}\right)$

Note: Award ***A1*** for each correct value. They may be embedded or stated explicitly.

OR

$(2x+1)(2x-9)=4\left(x^2-4x-\frac{9}{4}\right)$

$b=-4, c=-\frac{9}{4}$ ***A1A1***

Note: Award ***A1*** for each correct value. They must be stated explicitly.

METHOD 2

$-\frac{b}{2}=2$ OR $4+b=0 \Rightarrow b=-4$ ***A1***

attempt to form a valid equation to find c using their b ***(M1)***

$\left(-\frac{1}{2}\right)^2+-4\left(-\frac{1}{2}\right)+c=0$ OR $\left(\frac{9}{2}\right)^2+-4\left(\frac{9}{2}\right)+c=0$

$c=-\frac{9}{4}$ ***A1***

METHOD 3

attempt to form two valid equations in b and c ***(M1)***

$\left(-\frac{1}{2}\right)^2+b\left(-\frac{1}{2}\right)+c=0,\ \left(\frac{9}{2}\right)^2+b\left(\frac{9}{2}\right)+c=0$

$b=-4, c=-\frac{9}{4}$ ***A1A1***

METHOD 4

attempt to write $g(x)$ in the form $(x-h)^2+k$ and substitute for $x,\ h$ and $g(x)$ ***(M1)***

$\left(-\frac{1}{2}-2\right)^2+k=0 \Rightarrow k=-\frac{25}{4}$

$(x-2)^2-\frac{25}{4}$

$=x^2-4x-\frac{9}{4}$ ***A1A1***

$\left(b=-4, c=-\frac{9}{4}\right)$

Note: Award *A1* for each correct value. They may be embedded or stated explicitly.

[3 marks]

(e) Find the y-coordinate of the vertex of the graph of $y = g(x)$. [2]

Markscheme

attempt to substitute $x = 2$ into their $g(x)$ OR

complete the square on their $g(x)$ (may be seen in part (d)) *(M1)*

$y = -\frac{25}{4}$ *A1*

[2 marks]

(f) Find the product of the solutions of the equation $f(x) = g(x)$. [4]

Markscheme

$\frac{4x+2}{x-2} = \left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right)$ OR $\frac{4x+2}{x-2} = x^2 - 4x - \frac{9}{4}$

attempt to form a cubic equation *(M1)*

EITHER

$4x + 2 = (x - 2)\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right)$ OR $4x + 2 = \left(x^2 - 4x - \frac{9}{4}\right)(x - 2)$ OR

$(x - 2)\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right) - 4x - 2$ OR $(x - 2)\left(x^2 - 4x - \frac{9}{4}\right) - 4x - 2$

$x^3 + \ldots + \frac{5}{2}(= 0)$ OR $4x^3 + \ldots + 10(= 0)$ *(A1)(A1)*

Note: Award *(A1)* for each of the terms x^3 and $\frac{5}{2}$ or $4x^3$ and 10. Ignore extra terms.

product of roots $= \left(\frac{(-1)^3 \times \frac{5}{2}}{1}\right)$ OR $\left(\frac{(-1)^3 \times 10}{4}\right)$

$= -\frac{5}{2}$ ***A1***

OR

$4\left(x+\frac{1}{2}\right) = (x-2)\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$

$x = -\frac{1}{2}$ ***(A1)***

or $4 = x^2 + \ldots + 9 \Rightarrow x^2 + \ldots + 5 = 0$

product of roots of quadratic is 5 ***(A1)***

product is therefore $-\frac{1}{2} \times 5$

$= -\frac{5}{2}$ ***A1***

[4 marks]

4. [Maximum mark: 5]

Solve $3 \times 9^x + 5 \times 3^x - 2 = 0$. [5]

Markscheme

recognising a quadratic in 3^x ***(M1)***

$3 \times (3^x)^2 + 5 \times 3^x - 2 = 0$

valid attempt to solve a quadratic equation (factorising, use of formula, completing square, or otherwise) ***(M1)***

$(3 \times 3^x - 1)(3^x + 2) = 0$ OR $3^x = \frac{-5 \pm \sqrt{25+24}}{6}$ (or equivalent) ***(A1)***

$3^x = \frac{1}{3}$ (or $3^x = -2$) ***(A1)***

$x = -1$ ***A1***

Note: Award the final ***A1*** if candidate's answer includes $x = -1$ and $x = \log_3(-2)$. Award ***A0*** if other incorrect answers are given.

[5 marks]

5. [Maximum mark: 7]

A function $g(x)$ is defined by $g(x) = 2x^3 - 7x^2 + dx - e$, where $d,\ e \in \mathbb{R}$.

$\alpha,\ \beta$ and γ are the three roots of the equation $g(x) = 0$ where $\alpha,\ \beta,\ \gamma \in \mathbb{R}$.

(a) Write down the value of $\alpha + \beta + \gamma$. [1]

Markscheme

$\alpha + \beta + \gamma = \frac{7}{2}$ ***A1***

[1 mark]

A function $h(z)$ is defined by $h(z) = 2z^5 - 11z^4 + rz^3 + sz^2 + tz - 20$, where $r,\ s,\ t \in \mathbb{R}$.

$\alpha,\ \beta$ and γ are also roots of the equation $h(z) = 0$.

It is given that $h(z) = 0$ is satisfied by the complex number $z = p + 3\mathrm{i}$.

(b) Show that $p = 1$. [3]

Markscheme

$p - 3\mathrm{i}$ is also a root (seen anywhere) ***A1***

recognition of 5 roots and attempt to sum these roots ***(M1)***

$p + 3\mathrm{i} + p - 3\mathrm{i} + \frac{7}{2}$

$p + 3\mathrm{i} + p - 3\mathrm{i} + \frac{7}{2} = \frac{11}{2}$ ***A1***

$p = 1$ ***AG***

[3 marks]

It is now given that $h\left(\frac{1}{2}\right) = 0$, and $\alpha,\ \beta \in \mathbb{Z}^+,\ \alpha < \beta$ and $\gamma \in \mathbb{Q}$.

(c.i) Find the value of the product $\alpha\beta$. [2]

Markscheme

attempt to find product of 5 roots and equate to ± 10 ***(M1)***

$(1 + 3\text{i})(1 - 3\text{i})\frac{1}{2}\alpha\beta = 10$

$\alpha\beta = 2$ ***A1***

[2 marks]

(c.ii) Write down the value of α and the value of β. [1]

Markscheme

$\alpha = 1$ and $\beta = 2$ ***A1***

[1 mark]

6. [Maximum mark: 5]

Differentiate from first principles the function $f\left(x\right) = 3x^3 - x$. [5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

METHOD 1

$\frac{f(x+h)-f(x)}{h}$

$= \frac{\left(3(x+h)^3-(x+h)\right)-\left(3x^3-x\right)}{h}$ ***M1***

$= \frac{3\left(x^3+3x^2h+3xh^2+h^3\right)-x-h-3x^3+x}{h}$ ***(A1)***

$= \frac{9x^2h+9xh^2+3h^3-h}{h}$ ***A1***

cancelling h ***M1***

$= 9x^2 + 9xh + 3h^2 - 1$

then $\lim\limits_{h \to 0} \left(9x^2 + 9xh + 3h^2 - 1\right)$

$= 9x^2 - 1$ ***A1***

Note: Final ***A1*** dependent on all previous marks.

METHOD 2

$\frac{f(x+h)-f(x)}{h}$

$= \frac{\left(3(x+h)^3-(x+h)\right)-\left(3x^3-x\right)}{h}$ ***M1***

$= \frac{3\left((x+h)^3-x^3\right)+(x-(x+h))}{h}$ ***(A1)***

$= \frac{3h\left((x+h)^2+x(x+h)+x^2\right)-h}{h}$ ***A1***

cancelling h ***M1***

$= 3\left((x+h)^2 + x(x+h) + x^2\right) - 1$

then $\lim\limits_{h \to 0} \left(3\left((x+h)^2 + x(x+h) + x^2\right) - 1\right)$

$= 9x^2 - 1$ ***A1***

Note: Final ***A1*** dependent on all previous marks.

[5 marks]

7. [Maximum mark: 6]

Let $P(x) = 2x^4 - 15x^3 + ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$

(a) Given that $(x - 5)$ is a factor of $P(x)$, find a relationship between a, b and c. [2]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to substitute $x = 5$ and set equal to zero, or use of long / synthetic division ***(M1)***

$2 \times 5^4 - 15 \times 5^3 + a \times 5^2 + 5b + c = 0$ ***A1***

$(\Rightarrow 25a + 5b + c = 625)$

[2 marks]

(b) Given that $(x - 5)^2$ is a factor of $P(x)$, write down the value of $P'(5)$. [1]

Markscheme

0 ***A1***

[1 mark]

(c) Given that $(x - 5)^2$ is a factor of $P(x)$, and that $a = 2$, find the values of b and c. [3]

Markscheme

EITHER

attempt to solve $P'(5) = 0$ ***(M1)***

$\Rightarrow 8 \times 5^3 - 45 \times 5^2 + 4 \times 5 + b = 0$

OR

$\left(x^2-10x+25\right)\left(2x^2+\alpha x+\beta\right)=2x^4-15x^3+2x^2+bx+c$ ***(M1)***

comparing coefficients gives $\alpha=5, \beta=2$

THEN

$b=105$ ***A1***

$\therefore c=625-25\times 2-525$

$c=50$ ***A1***

[3 marks]

8. [Maximum mark: 7]

The lengths of two of the sides in a triangle are 4 cm and 5 cm. Let θ be the angle between the two given sides. The triangle has an area of $\frac{5\sqrt{15}}{2}$ cm^2.

(a) Show that $\sin\theta=\frac{\sqrt{15}}{4}$. [1]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

EITHER

$\frac{5\sqrt{15}}{2}=\frac{1}{2}\times 4\times 5\sin\theta$ ***A1***

OR

height of triangle is $\frac{5\sqrt{15}}{4}$ if using 4 as the base or $\sqrt{15}$ if using 5 as the base ***A1***

THEN

$\sin\theta=\frac{\sqrt{15}}{4}$ ***AG***

[1 mark]

(b) Find the two possible values for the length of the third side. [6]

Markscheme

let the third side be x

$x^2 = 4^2 + 5^2 - 2 \times 4 \times 5 \times \cos\theta$ ***M1***

valid attempt to find $\cos\theta$ ***(M1)***

Note: Do not accept writing $\cos\left(\arcsin\left(\frac{\sqrt{15}}{4}\right)\right)$ as a valid method.

$\cos\theta = \pm\sqrt{1 - \frac{15}{16}}$

$= \frac{1}{4}, \ -\frac{1}{4}$ ***A1A1***

$x^2 = 16 + 25 - 2 \times 4 \times 5 \times \pm\frac{1}{4}$

$x = \sqrt{31}$ or $\sqrt{51}$ ***A1A1***

[6 marks]

9. [Maximum mark: 7]

Solve the simultaneous equations

$\log_2 6x = 1 + 2\log_2 y$

$1 + \log_6 x = \log_6(15y - 25).$ [7]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

use of at least one "log rule" applied correctly for the first equation ***M1***

$\log_2 6x = \log_2 2 + 2\log_2 y$

$= \log_2 2 + \log_2 y^2$

$= \log_2 (2y^2)$

$\Rightarrow 6x = 2y^2$ ***A1***

use of at least one "log rule" applied correctly for the second equation ***M1***

$\log_6 (15y - 25) = 1 + \log_6 x$

$= \log_6 6 + \log_6 x$

$= \log_6 6x$

$\Rightarrow 15y - 25 = 6x$ ***A1***

attempt to eliminate x (or y) from their two equations ***M1***

$2y^2 = 15y - 25$

$2y^2 - 15y + 25 = 0$

$(2y - 5)(y - 5) = 0$

$x = \frac{25}{12}, y = \frac{5}{2},$ ***A1***

or $x = \frac{25}{3}, y = 5$ ***A1***

Note: x, y values do not have to be "paired" to gain either of the final two ***A*** marks.

[7 marks]

10. [Maximum mark: 7]

The curve C is given by the equation $y = x \tan\left(\frac{\pi x y}{4}\right)$.

(a) At the point (1, 1) , show that $\frac{dy}{dx} = \frac{2+\pi}{2-\pi}$. [5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to differentiate implicitly ***M1***

$\frac{dy}{dx} = x\sec^2\left(\frac{\pi xy}{4}\right)\left[\left(\frac{\pi}{4}x\frac{dy}{dx} + \frac{\pi}{4}y\right)\right] + \tan\left(\frac{\pi xy}{4}\right)$ ***A1A1***

Note: Award ***A1*** for each term.

attempt to substitute $x = 1, y = 1$ into their equation for $\frac{dy}{dx}$ ***M1***

$\frac{dy}{dx} = \frac{\pi}{2}\frac{dy}{dx} + \frac{\pi}{2} + 1$

$\frac{dy}{dx}\left(1 - \frac{\pi}{2}\right) = \frac{\pi}{2} + 1$ ***A1***

$\frac{dy}{dx} = \frac{2+\pi}{2-\pi}$ ***AG***

[5 marks]

(b) Hence find the equation of the normal to C at the point (1, 1). [2]

Markscheme

attempt to use gradient of normal $= \frac{-1}{\frac{dy}{dx}}$ ***(M1)***

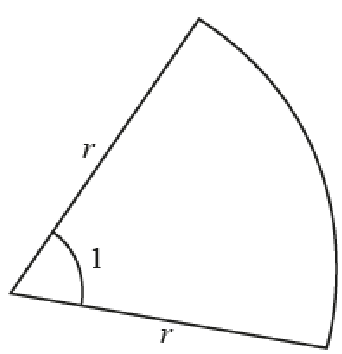
$= \frac{\pi-2}{\pi+2}$

so equation of normal is $y - 1 = \frac{\pi-2}{\pi+2}(x - 1)$ or $y = \frac{\pi-2}{\pi+2}x + \frac{4}{\pi+2}$ ***A1***

[2 marks]

11. [Maximum mark: 4]

A sector of a circle with radius r cm , where r > 0, is shown on the following diagram.
The sector has an angle of 1 radian at the centre.



Let the area of the sector be A cm^2 and the perimeter be P cm. Given that $A = P$, find the value of r. [4]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$A = P$

use of the correct formula for area and arc length ***(M1)***

perimeter is $r\theta + 2r$ ***(A1)***

Note: ***A1*** independent of previous ***M1***.

$\frac{1}{2}r^2(1) = r(1) + 2r$ ***A1***

$r^2 - 6r = 0$

$r = 6$ (as $r > 0$) ***A1***

Note: Do not award final ***A1*** if $r = 0$ is included.

[4 marks]

12. [Maximum mark: 5]

Find the equation of the tangent to the curve $y = \mathrm{e}^{2x} – 3x$ at the point where $x = 0$. [5]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$(x = 0 \Rightarrow) y = 1$ ***(A1)***

appreciate the need to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\left(\frac{\mathrm{d}y}{\mathrm{d}x} =\right) 2\mathrm{e}^{2x} - 3$ ***A1***

$(x = 0 \Rightarrow) \frac{\mathrm{d}y}{\mathrm{d}x} = -1$ ***A1***

$\frac{y-1}{x-0} = -1 \quad (y = 1 - x)$ ***A1***

[5 marks]

13. [Maximum mark: 7]

A and B are acute angles such that $\cos A = \frac{2}{3}$ and $\sin B = \frac{1}{3}$.

Show that $\cos(2A + B) = -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$. [7]

Markscheme

attempt to use $\cos(2A + B) = \cos 2A \cos B - \sin 2A \sin B$ (may be seen later) ***M1***

attempt to use any double angle formulae (seen anywhere) ***M1***

attempt to find either $\sin A$ or $\cos B$ (seen anywhere) ***M1***

$\cos A = \frac{2}{3} \Rightarrow \sin A \left(= \sqrt{1 - \frac{4}{9}}\right) = \frac{\sqrt{5}}{3}$ ***(A1)***

$\sin B = \frac{1}{3} \Rightarrow \cos B \left(= \sqrt{1 - \frac{1}{9}} = \frac{\sqrt{8}}{3}\right) = \frac{2\sqrt{2}}{3}$ ***A1***

$\cos 2A \left(= 2\cos^2 A - 1\right) = -\frac{1}{9}$ ***A1***

$\sin 2A \left(= 2\sin A \cos A\right) = \frac{4\sqrt{5}}{9}$ ***A1***

So $\cos(2A + B) = \left(-\frac{1}{9}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{4\sqrt{5}}{9}\right)\left(\frac{1}{3}\right)$

$= -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$ ***AG***

[7 marks]

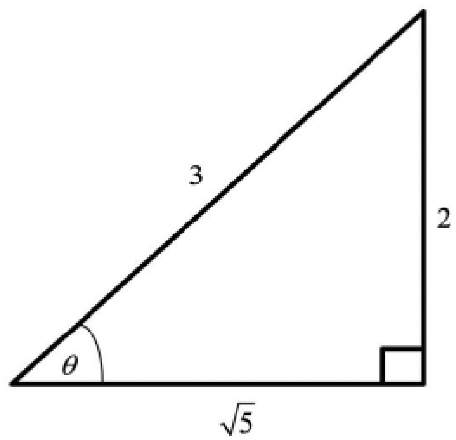
14. [Maximum mark: 4]

It is given that $\operatorname{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot\theta$. [4]

Markscheme

METHOD 1

attempt to use a right angled triangle ***M1***



correct placement of all three values and θ seen in the triangle ***(A1)***

$\cot\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer
The ***R1*** should be awarded independently for a negative value only given as a final answer.

METHOD 2

Attempt to use $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ ***M1***

$1 + \cot^2\theta = \frac{9}{4}$

$\cot^2\theta = \frac{5}{4}$ ***(A1)***

$\cot\theta = \pm\frac{\sqrt{5}}{2}$

$\cot\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

METHOD 3

$\sin\theta = \frac{2}{3}$

attempt to use $\sin^2\theta + \cos^2\theta = 1$ ***M1***

$\frac{4}{9} + \cos^2\theta = 1$

$\cos^2\theta = \frac{5}{9}$ ***(A1)***

$\cos\theta = \pm\frac{\sqrt{5}}{3}$

$\cos\theta < 0$ (since $\operatorname{cosec}\theta > 0$ puts θ in the second quadrant) ***R1***

$\cos\theta = -\frac{\sqrt{5}}{3}$

$\cot\theta = -\frac{\sqrt{5}}{2}$ ***A1***

Note: Award ***M1A1R0A0*** for $\cot\theta = \frac{\sqrt{5}}{2}$ seen as the final answer

The ***R1*** should be awarded independently for a negative value only given as a final answer.

[4 marks]

15. [Maximum mark: 8]

Consider the quartic equation $z^4 + 4z^3 + 8z^2 + 80z + 400 = 0,\ z \in \mathbb{C}$.

Two of the roots of this equation are $a + b\mathrm{i}$ and $b + a\mathrm{i}$, where $a,\ b \in \mathbb{Z}$.

Find the possible values of a. [8]

Markscheme

METHOD 1

other two roots are $a - b\mathrm{i}$ and $b - a\mathrm{i}$ ***A1***

sum of roots $= -4$ and product of roots $= 400$ ***A1***

attempt to set sum of four roots equal to -4 or 4 OR
attempt to set product of four roots equal to 400 ***M1***

$a + b\mathrm{i} + a - b\mathrm{i} + b + a\mathrm{i} + b - ai = -4$

$2a + 2b = -4 (\Rightarrow a + b = -2)$ ***A1***

$(a + b\mathrm{i})(a - b\mathrm{i})\ (b + a\mathrm{i})(b - a\mathrm{i}) = 400$

$\left(a^2 + b^2\right)^2 = 400$ ***A1***

$a^2 + b^2 = 20$

attempt to solve simultaneous equations ***(M1)***

$a = 2$ or $a = -4$ ***A1A1***

METHOD 2

other two roots are $a - b\mathrm{i}$ and $b - a\mathrm{i}$ ***A1***

$(z - (a + b\mathrm{i}))(z - (a - b\mathrm{i}))(z - (b + a\mathrm{i}))(z - (b - a\mathrm{i}))(= 0)$ ***A1***

$\left((z - a)^2 + b^2\right)\left((z - b)^2 + a^2\right)(= 0)$

$\left(z^2 - 2az + a^2 + b^2\right)\left(z^2 - 2bz + b^2 + a^2\right)(= 0)$ ***A1***

Attempt to equate coefficient of z^3 and constant with the given quartic equation ***M1***

$-2a - 2b = 4$ and $\left(a^2 + b^2\right)^2 = 400$ ***A1***

attempt to solve simultaneous equations ***(M1)***

$a = 2$ or $a = -4$ ***A1A1***

[8 marks]

16. [Maximum mark: 7]

Solve the equation $2\cos^2 x + 5\sin x = 4,\ 0 \leq x \leq 2\pi$. [7]

Markscheme

attempt to use $\cos^2 x = 1 - \sin^2 x$ ***M1***

$2\sin^2 x - 5\sin x + 2 = 0$ ***A1***

EITHER

attempting to factorise ***M1***

$(2\sin x - 1)(\sin x - 2)$ ***A1***

OR

attempting to use the quadratic formula ***M1***

$\sin x = \frac{5 \pm \sqrt{5^2 - 4 \times 2 \times 2}}{4} \left(= \frac{5 \pm 3}{4}\right)$ ***A1***

THEN

$\sin x = \frac{1}{2}$ ***(A1)***

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ ***A1A1***

[7 marks]

17. [Maximum mark: 5]

The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots $\alpha,\ \beta$ and $\alpha + \beta$.

Given that $\alpha\beta = -\frac{k^2}{4}$, find the value of k. [5]

Markscheme

$\alpha + \beta + \alpha + \beta = k$ ***(A1)***

$\alpha + \beta = \frac{k}{2}$

$\alpha\beta(\alpha + \beta) = -3k$ ***(A1)***

$\left(-\frac{k^2}{4}\right)\left(\frac{k}{2}\right) = -3k\ \left(-\frac{k^3}{8} = -3k\right)$ ***M1***

attempting to solve $-\frac{k^3}{8} + 3k = 0$ (or equivalent) for k ***(M1)***

$k = 2\sqrt{6}\ \left(= \sqrt{24}\right)(k > 0)$ ***A1***

Note: Award ***A0*** for $k = \pm 2\sqrt{6}\ \left(\pm\sqrt{24}\right)$.

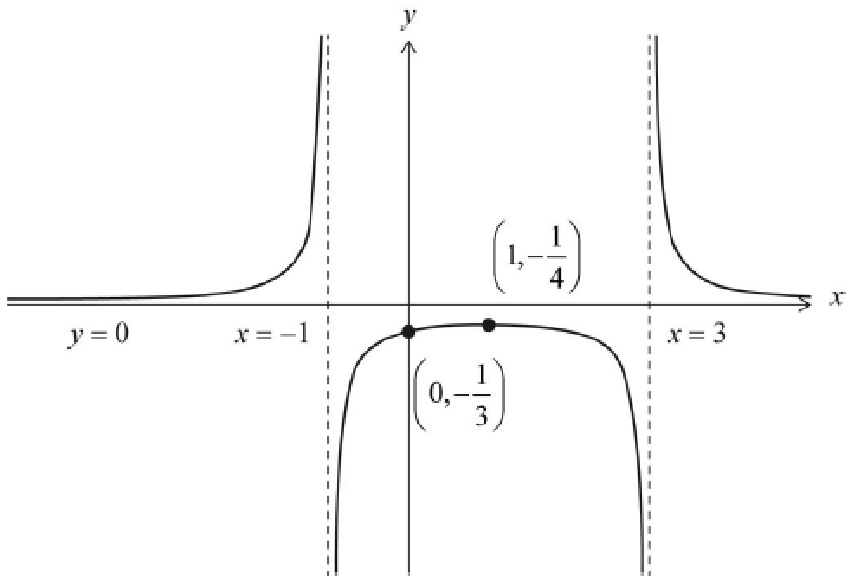
[5 marks]

18. [Maximum mark: 20]

A function f is defined by $f(x) = \frac{1}{x^2 - 2x - 3}$, where $x \in \mathbb{R},\ x \neq -1,\ x \neq 3$.

(a) Sketch the curve $y = f(x)$, clearly indicating any asymptotes with their equations. State the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [6]

Markscheme



y-intercept $\left(0, -\frac{1}{3}\right)$ ***A1***

Note: Accept an indication of $-\frac{1}{3}$ on the y-axis.

vertical asymptotes $x = -1$ and $x = 3$ ***A1***

horizontal asymptote $y = 0$ ***A1***

uses a valid method to find the x-coordinate of the local maximum point ***(M1)***

Note: For example, uses the axis of symmetry or attempts to solve $f\prime(x) = 0$.

local maximum point $\left(1, -\frac{1}{4}\right)$ ***A1***

Note: Award ***(M1)A0*** for a local maximum point at $x = 1$ and coordinates not given.

three correct branches with correct asymptotic behaviour and the key features in approximately correct relative positions to each other ***A1***

[6 marks]

A function g is defined by $g(x) = \frac{1}{x^2-2x-3}$, where $x \in \mathbb{R},\ x > 3$.

The inverse of g is g^{-1}.

(b.i) Show that $g^{-1}(x) = 1 + \frac{\sqrt{4x^2+x}}{x}$. [6]

Markscheme

$x = \frac{1}{y^2-2y-3}$ ***M1***

Note: Award ***M1*** for interchanging x and y (this can be done at a later stage).

EITHER

attempts to complete the square ***M1***

$y^2 - 2y - 3 = (y-1)^2 - 4$ ***A1***

$x = \frac{1}{(y-1)^2-4}$

$(y-1)^2 - 4 = \frac{1}{x} \left((y-1)^2 = 4 + \frac{1}{x}\right)$ ***A1***

$y - 1 = \pm\sqrt{4 + \frac{1}{x}} \left(= \pm\sqrt{\frac{4x+1}{x}}\right)$

OR

attempts to solve $xy^2 - 2xy - 3x - 1 = 0$ for y ***M1***

$y = \frac{-(-2x)\pm\sqrt{(-2x)^2+4x(3x+1)}}{2x}$ ***A1***

Note: Award ***A1*** even if − (in $\pm$) is missing

$= \frac{2x \pm \sqrt{16x^2+4x}}{2x}$ ***A1***

THEN

$= 1 \pm \frac{\sqrt{4x^2+x}}{x}$ ***A1***

$y > 3$ and hence $y = 1 - \frac{\sqrt{4x^2+x}}{x}$ is rejected ***R1***

Note: Award ***R1*** for concluding that the expression for y must have the '$+$' sign.
The ***R1*** may be awarded earlier for using the condition $x > 3$.

$y = 1 + \frac{\sqrt{4x^2+x}}{x}$

$g^{-1}(x) = 1 + \frac{\sqrt{4x^2+x}}{x}$ ***AG***

[6 marks]

(b.ii) State the domain of g^{-1}. [1]

Markscheme

domain of g^{-1} is $x > 0$ ***A1***

[1 mark]

A function h is defined by $h(x) = \arctan \frac{x}{2}$, where $x \in \mathbb{R}$.

(c) Given that $(h \circ g)(a) = \frac{\pi}{4}$, find the value of a.

Give your answer in the form $p + \frac{q}{2}\sqrt{r}$, where $p,\ q,\ r \in \mathbb{Z}^+$. [7]

Markscheme

attempts to find $(h \circ g)(a)$ ***(M1)***

$(h \circ g)(a) = \arctan\left(\frac{g(a)}{2}\right) \quad \left((h \circ g)(a) = \arctan\left(\frac{1}{2(a^2-2a-3)}\right)\right)$ ***(A1)***

$\arctan\left(\frac{g(a)}{2}\right) = \frac{\pi}{4} \quad \left(\arctan\left(\frac{1}{2(a^2-2a-3)}\right) = \frac{\pi}{4}\right)$

attempts to solve for $g(a)$ ***M1***

$\Rightarrow g(a) = 2 \quad \left(\frac{1}{(a^2-2a-3)} = 2\right)$

EITHER

$\Rightarrow a = g^{-1}(2)$ ***A1***

attempts to find their $g^{-1}(2)$ ***M1***

$a = 1 + \frac{\sqrt{4(2)^2+2}}{2}$ ***A1***

Note: Award all available marks to this stage if x is used instead of a.

OR

$\Rightarrow 2a^2 - 4a - 7 = 0$ ***A1***

attempts to solve their quadratic equation ***M1***

$a = \frac{-(-4) \pm \sqrt{(-4)^2 + 4(2)(7)}}{4} \quad \left(= \frac{4 \pm \sqrt{72}}{4}\right)$ ***A1***

Note: Award all available marks to this stage if x is used instead of a.

THEN

$a = 1 + \frac{3}{2}\sqrt{2}$ (as $a > 3$) ***A1***

$(p = 1,\ q = 3,\ r = 2)$

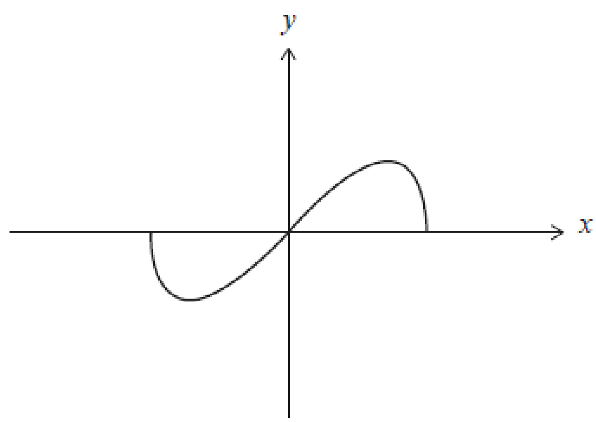
Note: Award ***A1*** for $a = 1 + \frac{1}{2}\sqrt{18}\ (p = 1,\ q = 1,\ r = 18)$

[7 marks]

19. [Maximum mark: 8]

A function f is defined by $f(x) = x\sqrt{1 - x^2}$ where $-1 \leq x \leq 1$.

The graph of $y = f(x)$ is shown below.



(a) Show that f is an odd function. [2]

Markscheme

attempts to replace x with $-x$ ***M1***

$f(-x) = -x\sqrt{1 - (-x)^2}$

$= -x\sqrt{1 - (-x)^2}(= -f(x))$ ***A1***

Note: Award ***M1A1*** for an attempt to calculate both $f(-x)$ and $-f(-x)$ independently, showing that they are equal.

Note: Award ***M1A0*** for a graphical approach including evidence that **either** the graph is invariant after rotation by 180° about the origin **or** the graph is invariant after a reflection in the y-axis and then in the x-axis (or vice versa).

so f is an odd function ***AG***

[2 marks]

(b) The range of f is $a \leq y \leq b$, where $a,\ b \in \mathbb{R}$.

Find the value of a and the value of b. [6]

Markscheme

attempts both product rule and chain rule differentiation to find $f\prime(x)$ ***M1***

$f\prime(x) = x \times \frac{1}{2} \times (-2x) \times \left(1 - x^2\right)^{-\frac{1}{2}} + \left(1 - x^2\right)^{\frac{1}{2}} \times 1 \left(= \sqrt{1 - x^2} - \frac{x^2}{\sqrt{1-x^2}}\right)$ ***A1***

$= \frac{1-2x^2}{\sqrt{1-x^2}}$

sets their $f\prime(x) = 0$ ***M1***

$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$ ***A1***

attempts to find at least one of $f\left(\pm\frac{1}{\sqrt{2}}\right)$ ***(M1)***

Note: Award **M1** for an attempt to evaluate $f(x)$ at least at one of their $f\prime(x) = 0$ roots.

$a = -\frac{1}{2}$ and $b = \frac{1}{2}$ ***A1***

Note: Award ***A1*** for $-\frac{1}{2} \leq y \leq \frac{1}{2}$.

[6 marks]

20. [Maximum mark: 19]

(a) Show that $\cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$. [1]

Markscheme

stating the relationship between cot and $\tan$ and stating the identity for $\tan 2\theta$ ***M1***

$\cot 2\theta = \frac{1}{\tan 2\theta}$ and $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$ ***AG***

[1 mark]

(b) Verify that $x = \tan\theta$ and $x = -\cot\theta$ satisfy the equation $x^2 + (2\cot 2\theta)x - 1 = 0$. [7]

Markscheme

METHOD 1

attempting to substitute $\tan\theta$ for x and using the result from (a) ***M1***

LHS = $\tan^2\theta + 2\tan\theta\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$\tan^2\theta + 1 - \tan^2\theta - 1 = 0$(= RHS) ***A1***

so $x = \tan\theta$ satisfies the equation ***AG***

attempting to substitute $-\cot\theta$ for x and using the result from (a) ***M1***

LHS = $\cot^2\theta - 2\cot\theta\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$= \frac{1}{\tan^2\theta} - \left(\frac{1-\tan^2\theta}{\tan^2\theta}\right) - 1$ ***A1***

$\frac{1}{\tan^2\theta} - \frac{1}{\tan^2\theta} + 1 - 1 = 0$(= RHS) ***A1***

so $x = -\cot\theta$ satisfies the equation ***AG***

METHOD 2

let $\alpha = \tan\theta$ and $\beta = -\cot\theta$

attempting to find the sum of roots ***M1***

$\alpha + \beta = \tan\theta - \frac{1}{\tan\theta}$

$= \frac{\tan^2\theta - 1}{\tan\theta}$ ***A1***

$= -2\cot 2\theta$ (from part (a)) ***A1***

attempting to find the product of roots ***M1***

$\alpha\beta = \tan\theta \times (-\cot\theta)$ ***A1***

$= -1$ ***A1***

the coefficient of x and the constant term in the quadratic are $2\cot 2\theta$ and -1 respectively ***R1***

hence the two roots are $\alpha = \tan\theta$ and $\beta = -\cot\theta$ ***AG***

[7 marks]

(c) Hence, or otherwise, show that the exact value of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. [5]

Markscheme

METHOD 1

$x = \tan\frac{\pi}{12}$ and $x = -\cot\frac{\pi}{12}$ are roots of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$ ***R1***

Note: Award ***R1*** if only $x = \tan\frac{\pi}{12}$ is stated as a root of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$.

$x^2 + 2\sqrt{3}x - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$x = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ $\left(-\cot\frac{\pi}{12} < 0\right)$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

METHOD 2

attempting to substitute $\theta = \frac{\pi}{12}$ into the identity for $\tan 2\theta$ ***M1***

$\tan\frac{\pi}{6} = \frac{2\tan\frac{\pi}{12}}{1-\tan^2\frac{\pi}{12}}$

$\tan^2\frac{\pi}{12} + 2\sqrt{3}\tan\frac{\pi}{12} - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$\tan\frac{\pi}{12} = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

[5 marks]

(d) Using the results from parts (b) and (c) find the exact value of $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$.

Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$. [6]

Markscheme

$\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$ is the sum of the roots of $x^2 + \left(2\cot\frac{\pi}{12}\right)x - 1 = 0$ ***R1***

$\tan\frac{\pi}{24} - \cot\frac{\pi}{24} = -2\cot\frac{\pi}{12}$ ***A1***

$= \frac{-2}{2-\sqrt{3}}$ ***A1***

attempting to rationalise **their** denominator ***(M1)***

$= -4 - 2\sqrt{3}$ ***A1A1***

[6 marks]

21. [Maximum mark: 8]

A function f is defined by $f(x) = \frac{2x-1}{x+1}$, where $x \in \mathbb{R},\ x \neq -1$.

The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

(a.i) Write down the equation of the vertical asymptote. [1]

Markscheme

$x = -1$ ***A1***

[1 mark]

(a.ii) Write down the equation of the horizontal asymptote. [1]

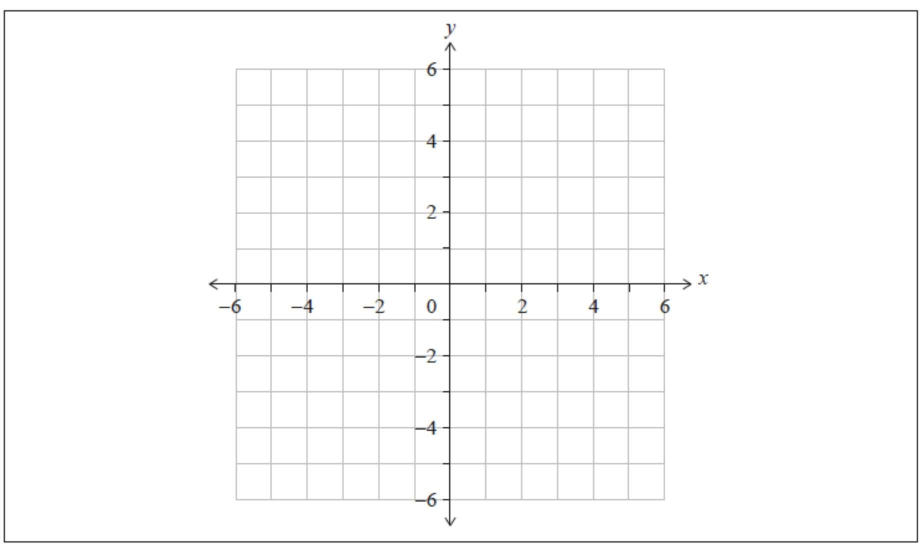
Markscheme

$y = 2$ ***A1***

[1 mark]

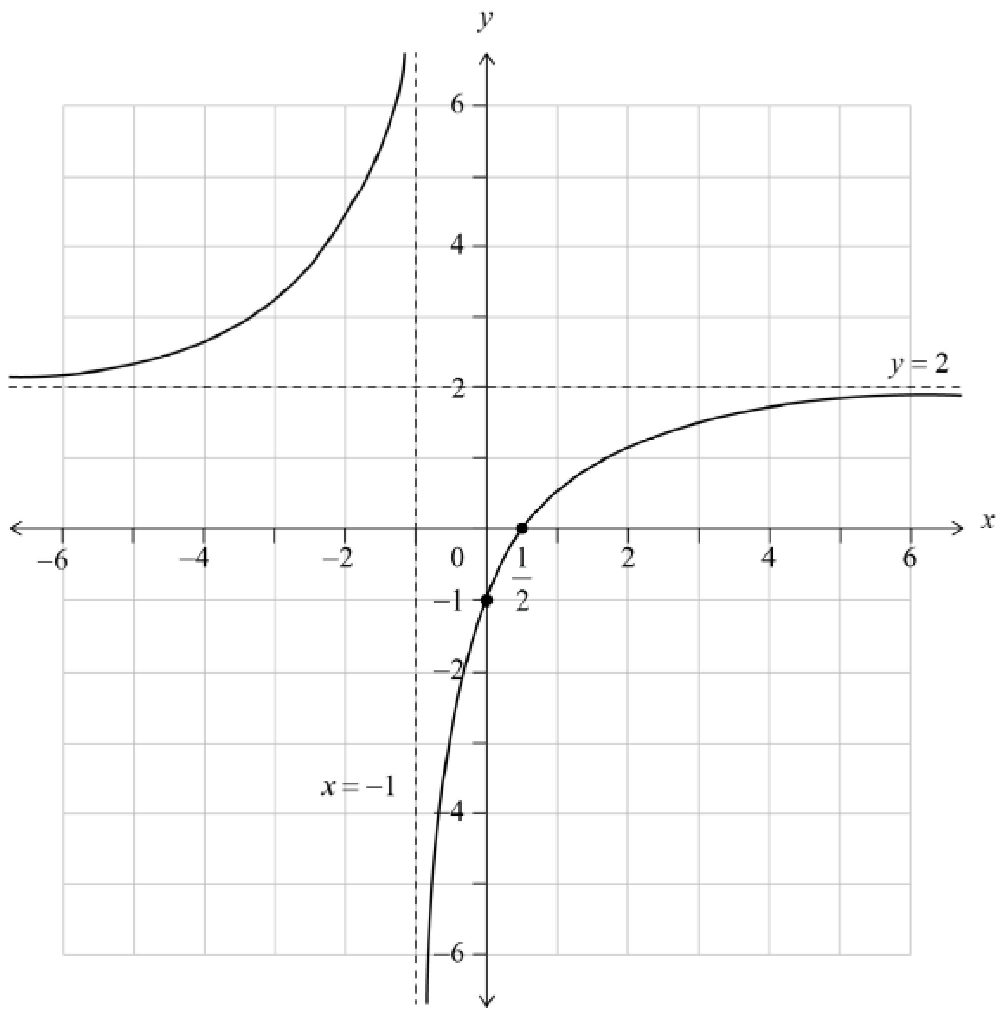
(b) On the set of axes below, sketch the graph of $y = f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes.



[3]

Markscheme



rational function shape with two branches in opposite quadrants, with two correctly positioned asymptotes and asymptotic behaviour shown ***A1***

axes intercepts clearly shown at $x = \frac{1}{2}$ and $y = -1$ ***A1A1***

[3 marks]

(c) Hence, solve the inequality $0 < \frac{2x-1}{x+1} < 2$. [1]

Markscheme

$x > \frac{1}{2}$ ***A1***

Note: Accept correct alternative correct notation, such as $\left(\frac{1}{2},\ \infty\right)$ and $\left]\frac{1}{2}, \infty\right[$.

[1 mark]

(d) Solve the inequality $0 < \frac{2|x|-1}{|x|+1} < 2$. [2]

Markscheme

EITHER

attempts to sketch $y = \frac{2|x|-1}{|x|+1}$ ***(M1)***

OR

attempts to solve $2|x| - 1 = 0$ ***(M1)***

Note: Award the ***(M1)*** if $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ are identified.

THEN

$x < -\frac{1}{2}$ or $x > \frac{1}{2}$ ***A1***

Note: Accept the use of a comma. Condone the use of 'and'. Accept correct alternative notation.

[2 marks]

22. [Maximum mark: 5]

Solve the equation $\log_3 \sqrt{x} = \frac{1}{2\log_2 3} + \log_3(4x^3)$, where $x > 0$. [5]

Markscheme

attempt to use change the base *(M1)*

$\log_3 \sqrt{x} = \frac{\log_3 2}{2} + \log_3(4x^3)$

attempt to use the power rule *(M1)*

$\log_3 \sqrt{x} = \log_3 \sqrt{2} + \log_3(4x^3)$

attempt to use product or quotient rule for logs, $\ln a + \ln b = \ln ab$ *(M1)*

$\log_3 \sqrt{x} = \log_3 \left(4\sqrt{2}x^3\right)$

Note: The ***M*** marks are for attempting to use the relevant log rule and may be applied in any order and at any time during the attempt seen.

$\sqrt{x} = 4\sqrt{2}x^3$

$x = 32x^6$

$x^5 = \frac{1}{32}$ *(A1)*

$x = \frac{1}{2}$ ***A1***

[5 marks]

23. [Maximum mark: 7]

The equation $3px^2 + 2px + 1 = p$ has two real, distinct roots.

(a) Find the possible values for p. [5]

Markscheme

attempt to use discriminant $b^2 - 4ac(> 0)$ ***M1***

$(2p)^2 - 4(3p)(1-p)(> 0)$

$16p^2 - 12p(> 0)$ *(A1)*

$p(4p - 3)(> 0)$

attempt to find critical values $\left(p = 0,\ p = \frac{3}{4}\right)$ ***M1***

recognition that discriminant > 0 *(M1)*

$p < 0$ or $p > \frac{3}{4}$ ***A1***

Note: Condone 'or' replaced with 'and', a comma, or no separator

[5 marks]

(b) Consider the case when $p = 4$. The roots of the equation can be expressed in the form $x = \frac{a \pm \sqrt{13}}{6}$, where $a \in \mathbb{Z}$. Find the value of a. [2]

Markscheme

$p = 4 \Rightarrow 12x^2 + 8x - 3 = 0$

valid attempt to use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (or equivalent) ***M1***

$x = \frac{-8 \pm \sqrt{208}}{24}$

$x = \frac{-2 \pm \sqrt{13}}{6}$

$a = -2$ ***A1***

[2 marks]

24. [Maximum mark: 7]

Consider the curve with equation $\left(x^2 + y^2\right)y^2 = 4x^2$ where $x \geq 0$ and $-2 < y < 2$.

Show that the curve has no local maximum or local minimum points for $x > 0$. [7]

Markscheme

attempt at implicit differentiation, including use of the product rule ***(M1)***

EITHER

$\left(2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)y^2 + \left(x^2 + y^2\right)2y\frac{\mathrm{d}y}{\mathrm{d}x} = 8x$ ***A1A1A1***

Note: Award ***A1*** for each of $\left(2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)y^2$, $\left(x^2 + y^2\right)2y\frac{\mathrm{d}y}{\mathrm{d}x}$ and $8x$

OR

$x^2y^2 + y^4 = 4x^2$

$2xy^2 + 2x^2y\frac{\mathrm{d}y}{\mathrm{d}x} + 4y^3\frac{\mathrm{d}y}{\mathrm{d}x} = 8x$ ***A1A1A1***

Note: Award ***A1*** for each of $2xy^2 + 2x^2y\frac{\mathrm{d}y}{\mathrm{d}x}$, $4y^3\frac{\mathrm{d}y}{\mathrm{d}x}$ and $8x$.

THEN

at a local maximum or minimum point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***(M1)***

$2xy^2 = 8x$

$x = 0$ or $y^2 = 4(\Rightarrow y = \pm 2)$ ***A1***

Note: Award ***A0*** for $x = 0$ or $y = 2$

since $x > 0$ and $-2 < y < 2$ there are no solutions ***R1***

hence there are no local maximum or minimum points ***AG***

[7 marks]

25. [Maximum mark: 7]

Consider the equation $z^4 + pz^3 + 54z^2 - 108z + 80 = 0$ where $z \in \mathbb{C}$ and $p \in \mathbb{R}$.

Three of the roots of the equation are $3 + \mathrm{i},\ \alpha$ and α^2, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all the roots of the equation, find the value of α. [4]

Markscheme

product of roots $= 80$ ***(A1)***

$3 - \mathrm{i}$ is a root ***(A1)***

attempt to set up an equation involving the product of their four roots and ± 80 ***(M1)***

$(3 + \mathrm{i})(3 - \mathrm{i})\alpha^3 = 80 \Rightarrow 10\alpha^3 = 80$

$\alpha = 2$ ***A1***

[4 marks]

(b) Find the value of p. [3]

Markscheme

METHOD 1

sum of roots $= -p$ ***(A1)***

$-p = 3 + \mathrm{i} + 3 - \mathrm{i} + 2 + 4$ ***(M1)***

Note: Accept $p = 3 + \mathrm{i} + 3 - \mathrm{i} + 2 + 4$ for ***(M1)***

$p = -12$ ***A1***

METHOD 2

$(z - (3 + \mathrm{i}))(z - (3 - \mathrm{i}))(z - 2)(z - 4)$ ***(M1)***

$((z-3)-\mathrm{i})((z-3)+\mathrm{i})(z-2)(z-4)$ ***(A1)***

$\left(z^2-6z+10\right)\left(z^2-6z+8\right)=z^4-12z^3+\ldots$

$p=-12$ ***A1***

[3 marks]