Revision (31.01) [210 marks]

1. [Maximum mark: 20]

The function f is defined by $f(x) = \cos^2 x - 3\sin^2 x, \quad 0 \leq x \leq \pi$.

(a) Find the roots of the equation $f(x) = 0$. [5]

(b.i) Find $f\prime(x)$. [2]

(b.ii) Hence find the coordinates of the points on the graph of $y = f(x)$ where $f\prime(x) = 0$. [5]

(c) Sketch the graph of $y = |f(x)|$, clearly showing the coordinates of any points where $f\prime(x) = 0$ and any points where the graph meets the coordinate axes. [4]

(d) Hence or otherwise, solve the inequality $|f(x)| > 1$. [4]

2. [Maximum mark: 9]

Let $f\,(x) = \frac{x^2-10x+5}{x+1},\ x \in \mathbb{R},\ x \neq -1$.

(a) Find the co-ordinates of all stationary points. [4]

(b) Write down the equation of the vertical asymptote. [1]

(c) With justification, state if each stationary point is a minimum, maximum or horizontal point of inflection. [4]

3. [Maximum mark: 16]

Consider the function $f(x) = \frac{4x+2}{x-2},\ x \neq 2$.

(a) Sketch the graph of $y = f(x)$. On your sketch, indicate the values of any axis intercepts and label any asymptotes with their equations. [5]

(b) Write down the range of f. [1]

Consider the function $g(x) = x^2 + bx + c$. The graph of g has an axis of symmetry at $x = 2$.

The two roots of $g(x) = 0$ are $-\frac{1}{2}$ and p, where $p \in \mathbb{Q}$.

(c) Show that $p = \frac{9}{2}$. [1]

(d) Find the value of b and the value of c. [3]

(e) Find the y-coordinate of the vertex of the graph of $y = g(x)$. [2]

(f) Find the product of the solutions of the equation $f(x) = g(x)$. [4]

4. [Maximum mark: 5]

Solve $3 \times 9^x + 5 \times 3^x - 2 = 0$. [5]

5. [Maximum mark: 7]

A function $g(x)$ is defined by $g(x) = 2x^3 - 7x^2 + dx - e$, where $d,\ e \in \mathbb{R}$.

$\alpha,\ \beta$ and γ are the three roots of the equation $g(x) = 0$ where $\alpha,\ \beta,\ \gamma \in \mathbb{R}$.

(a) Write down the value of $\alpha + \beta + \gamma$. [1]

A function $h(z)$ is defined by
$h(z) = 2z^5 - 11z^4 + rz^3 + sz^2 + tz - 20$, where $r,\ s,\ t \in \mathbb{R}$.

$\alpha,\ \beta$ and γ are also roots of the equation $h(z) = 0$.

It is given that $h(z) = 0$ is satisfied by the complex number $z = p + 3\mathrm{i}$.

(b) Show that $p = 1$. [3]

It is now given that $h\left(\frac{1}{2}\right) = 0$, and $\alpha,\ \beta \in \mathbb{Z}^+,\ \alpha < \beta$ and $\gamma \in \mathbb{Q}$.

(c.i) Find the value of the product $\alpha\beta$. [2]

(c.ii) Write down the value of α and the value of β. [1]

6. [Maximum mark: 5]
Differentiate from first principles the function $f\left(x\right) = 3x^3 - x$. [5]

7. [Maximum mark: 6]
Let $P\left(x\right) = 2x^4 - 15x^3 + ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$

(a) Given that $\left(x - 5\right)$ is a factor of $P\left(x\right)$, find a relationship between a, b and c. [2]

(b) Given that $\left(x - 5\right)^2$ is a factor of $P\left(x\right)$, write down the value of $P'\left(5\right)$. [1]

(c) Given that $\left(x - 5\right)^2$ is a factor of $P\left(x\right)$, and that $a = 2$, find the values of b and c. [3]

8. [Maximum mark: 7]

The lengths of two of the sides in a triangle are 4 cm and 5 cm. Let θ be the angle between the two given sides. The triangle has an area of $\frac{5\sqrt{15}}{2}$ cm^2.

(a) Show that $\sin\theta = \frac{\sqrt{15}}{4}$. [1]

(b) Find the two possible values for the length of the third side. [6]

9. [Maximum mark: 7]

Solve the simultaneous equations

$$\log_2 6x = 1 + 2\log_2 y$$

$$1 + \log_6 x = \log_6 (15y - 25).$$ [7]

10. [Maximum mark: 7]

The curve C is given by the equation $y = x\tan\left(\frac{\pi xy}{4}\right)$.

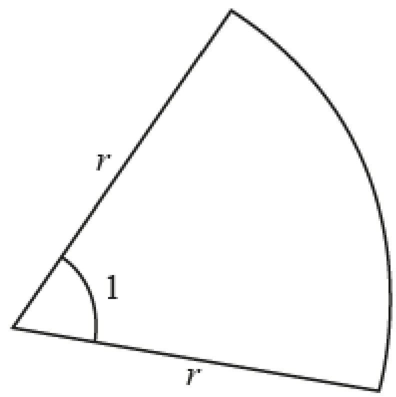
(a) At the point (1, 1) , show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2+\pi}{2-\pi}$. [5]

(b) Hence find the equation of the normal to C at the point (1, 1). [2]

11. [Maximum mark: 4]

A sector of a circle with radius r cm , where $r > 0$, is shown on the following diagram.

The sector has an angle of 1 radian at the centre.



[4]

Let the area of the sector be A cm^2 and the perimeter be P cm. Given that $A = P$, find the value of r.

12. [Maximum mark: 5]

Find the equation of the tangent to the curve $y = \mathrm{e}^{2x} – 3x$ at the point where $x = 0$. [5]

13. [Maximum mark: 7]

A and B are acute angles such that $\cos A = \frac{2}{3}$ and $\sin B = \frac{1}{3}$.

Show that $\cos\left(2A + B\right) = -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$. [7]

14. [Maximum mark: 4]

It is given that $\operatorname{cosec}\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $\cot\theta$. [4]

15. [Maximum mark: 8]

Consider the quartic equation
$z^4 + 4z^3 + 8z^2 + 80z + 400 = 0,\ z \in \mathbb{C}$.

Two of the roots of this equation are $a + b\mathrm{i}$ and $b + a\mathrm{i}$, where $a,\ b \in \mathbb{Z}$.

Find the possible values of a. [8]

16. [Maximum mark: 7]

Solve the equation $2\cos^2 x + 5\sin x = 4,\ 0 \le x \le 2\pi$. [7]

17. [Maximum mark: 5]

The cubic equation $x^3 - kx^2 + 3k = 0$ where $k > 0$ has roots $\alpha,\ \beta$ and $\alpha + \beta$.

Given that $\alpha\beta = -\frac{k^2}{4}$, find the value of k. [5]

18. [Maximum mark: 20]

A function f is defined by $f(x) = \frac{1}{x^2 - 2x - 3}$, where $x \in \mathbb{R},\ x \ne -1,\ x \ne 3$.

(a) Sketch the curve $y = f(x)$, clearly indicating any asymptotes with their equations. State the coordinates of any local maximum or minimum points and any points of intersection with the coordinate axes. [6]

A function g is defined by $g(x) = \frac{1}{x^2 - 2x - 3}$, where $x \in \mathbb{R},\ x > 3$.

The inverse of g is g^{-1}.

(b.i) Show that $g^{-1}(x) = 1 + \frac{\sqrt{4x^2+x}}{x}$. [6]

(b.ii) State the domain of g^{-1}. [1]

A function h is defined by $h(x) = \arctan\frac{x}{2}$, where $x \in \mathbb{R}$.

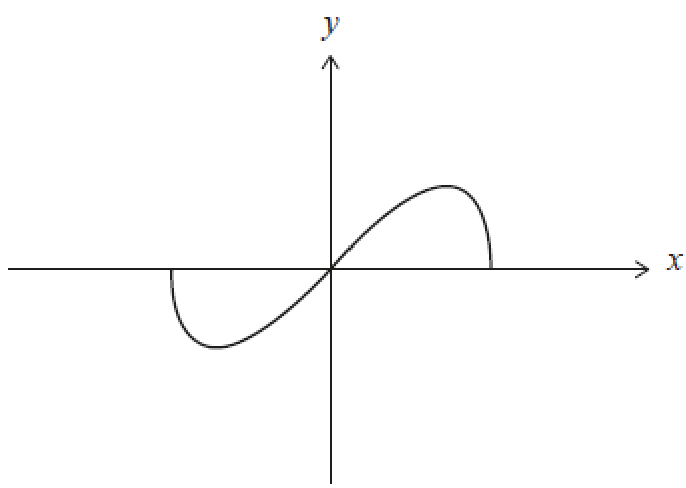
(c) Given that $(h \circ g)(a) = \frac{\pi}{4}$, find the value of a.

Give your answer in the form $p + \frac{q}{2}\sqrt{r}$, where $p,\ q,\ r \in \mathbb{Z}^+$. [7]

19. [Maximum mark: 8]

A function f is defined by $f(x) = x\sqrt{1-x^2}$ where $-1 \leq x \leq 1$.

The graph of $y = f(x)$ is shown below.



(a) Show that f is an odd function. [2]

(b) The range of f is $a \leq y \leq b$, where $a,\ b \in \mathbb{R}$.

Find the value of a and the value of b. [6]

20. [Maximum mark: 19]

(a) Show that $\cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$. [1]

(b) Verify that $x = \tan\theta$ and $x = -\cot\theta$ satisfy the equation $x^2 + (2\cot 2\theta)x - 1 = 0$. [7]

(c) Hence, or otherwise, show that the exact value of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. [5]

(d) Using the results from parts (b) and (c) find the exact value of $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$.

Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$. [6]

21. [Maximum mark: 8]

A function f is defined by $f(x) = \frac{2x-1}{x+1}$, where $x \in \mathbb{R},\ x \neq -1$.

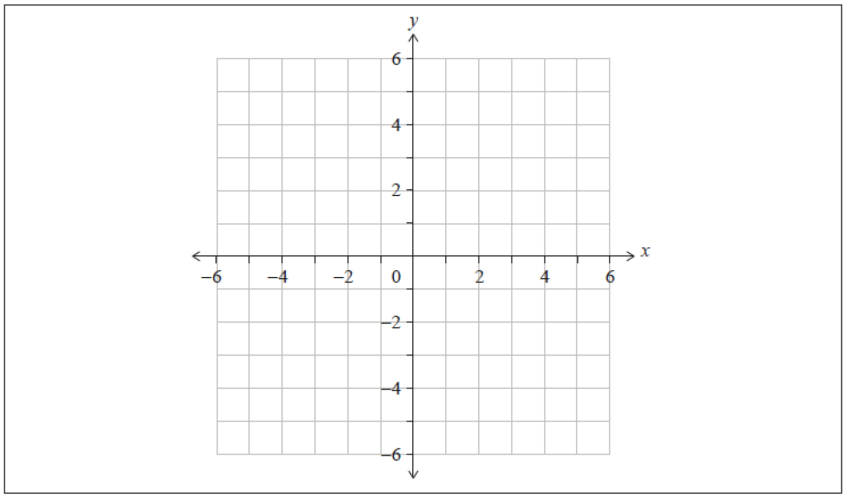
The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

(a.i) Write down the equation of the vertical asymptote. [1]

(a.ii) Write down the equation of the horizontal asymptote. [1]

(b) On the set of axes below, sketch the graph of $y = f(x)$.

On your sketch, clearly indicate the asymptotes and the position of any points of intersection with the axes.



[3]

(c) Hence, solve the inequality $0 < \frac{2x-1}{x+1} < 2$. [1]

(d) Solve the inequality $0 < \frac{2|x|-1}{|x|+1} < 2$. [2]

22. [Maximum mark: 5]

Solve the equation $\log_3 \sqrt{x} = \frac{1}{2\log_2 3} + \log_3\left(4x^3\right)$, where $x > 0$. [5]

23. [Maximum mark: 7]

The equation $3px^2 + 2px + 1 = p$ has two real, distinct roots.

(a) Find the possible values for p. [5]

(b) Consider the case when $p = 4$. The roots of the equation can be expressed in the form $x = \frac{a \pm \sqrt{13}}{6}$, where $a \in \mathbb{Z}$. Find the value of a. [2]

24. [Maximum mark: 7]

Consider the curve with equation $(x^2 + y^2)y^2 = 4x^2$ where $x \geq 0$ and $-2 < y < 2$.

Show that the curve has no local maximum or local minimum points for $x > 0$. [7]

25. [Maximum mark: 7]

Consider the equation $z^4 + pz^3 + 54z^2 - 108z + 80 = 0$ where $z \in \mathbb{C}$ and $p \in \mathbb{R}$.

Three of the roots of the equation are $3 + \mathrm{i}$, α and α^2, where $\alpha \in \mathbb{R}$.

(a) By considering the product of all the roots of the equation, find the value of α. [4]

(b) Find the value of p. [3]