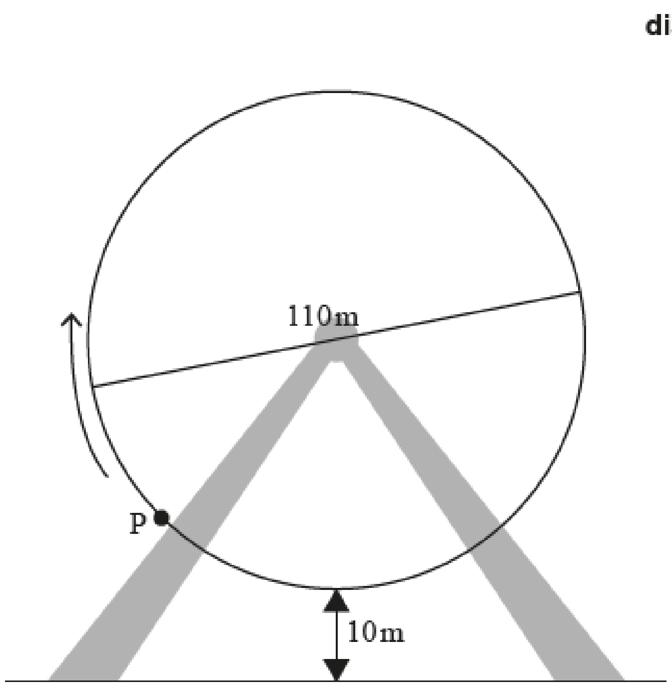
Trig modelling [55 marks]

1. [Maximum mark: 5]

A Ferris wheel with diameter 110 metres rotates at a constant speed. The lowest point on the wheel is 10 metres above the ground, as shown on the following diagram. P is a point on the wheel. The wheel starts moving with P at the lowest point and completes one revolution in 20 minutes.



The height, h metres, of P above the ground after t minutes is given by $h(t) = a\cos(bt) + c$, where $a,\ b,\ c \in \mathbb{R}$.

Find the values of a, b and c. [5]

Markscheme

amplitude is $\frac{110}{2} = 55$ ***(A1)***

$a = -55$ ***A1***

$c = 65$ ***A1***

$\frac{2\pi}{b} = 20$ OR $-55\cos(20b) + 65 = 10$ ***(M1)***

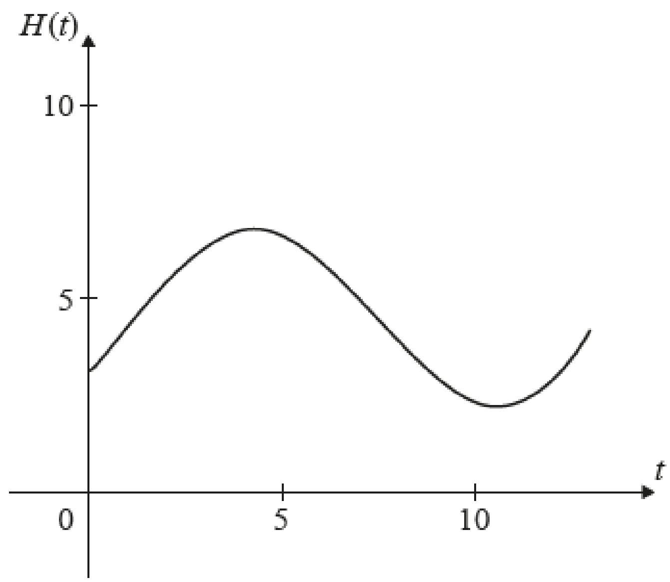
$b = \frac{\pi}{10} (= 0.314)$ ***A1***

[5 marks]

2. [Maximum mark: 15]

The height of water, in metres, in Dungeness harbour is modelled by the function $H(t) = a \sin(b(t - c)) + d$, where t is the number of hours after midnight, and $a,\ b,\ c$ and d are constants, where $a > 0,\ b > 0$ and $c > 0$.

The following graph shows the height of the water for 13 hours, starting at midnight.



The first high tide occurs at $04 : 30$ and the next high tide occurs 12 hours later. Throughout the day, the height of the water fluctuates between $2.2\,\mathrm{m}$ and $6.8\,\mathrm{m}$.

All heights are given correct to one decimal place.

(a) Show that $b = \frac{\pi}{6}$. [1]

Markscheme

$12 = \frac{2\pi}{b}$ OR $b = \frac{2\pi}{12}$ ***A1***

$b = \frac{\pi}{6}$ ***AG***

[1 mark]

(b) Find the value of a. [2]

Markscheme

$a = \frac{6.8-2.2}{2}$ OR $a = \frac{\max-\min}{2}$ ***(M1)***

$= 2.3\,(\mathrm{m})$ ***A1***

[2 marks]

(c) Find the value of d. [2]

Markscheme

$d = \frac{6.8+2.2}{2}$ OR $d = \frac{\max+\min}{2}$ ***(M1)***

$= 4.5\,(\mathrm{m})$ ***A1***

[2 marks]

(d) Find the smallest possible value of c. [3]

Markscheme

METHOD 1

substituting $t = 4.5$ and $H = 6.8$ for example into their equation for H ***(A1)***

$$6.8 = 2.3\sin\left(\frac{\pi}{6}(4.5 - c)\right) + 4.5$$

attempt to solve their equation ***(M1)***

$c = 1.5$ ***A1***

METHOD 2

using horizontal translation of $\frac{12}{4}$ ***(M1)***

$4.5 - c = 3$ ***(A1)***

$c = 1.5$ ***A1***

METHOD 3

$H\prime(t) = (2.3)\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}(t - c)\right)$ ***(A1)***

attempts to solve their $H\prime(4.5) = 0$ for c ***(M1)***

$$(2.3)\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}(4.5 - c)\right) = 0$$

$c = 1.5$ ***A1***

[3 marks]

(e) Find the height of the water at $12:00$. [2]

Markscheme

attempt to find H when $t = 12$ or $t = 0$, graphically or algebraically ***(M1)***

$H = 2.87365\ldots$

$H = 2.87\,(\mathrm{m})$ ***A1***

[2 marks]

(f) Determine the number of hours, over a 24-hour period, for which the tide is higher than 5 metres. [3]

Markscheme

attempt to solve $5 = 2.3\sin\left(\frac{\pi}{6}(t - 1.5)\right) + 4.5$ ***(M1)***

times are $t = 1.91852\ldots$ and $t = 7.08147\ldots\ ,\ (t = 13.9185\ldots,\ t = 19.0814\ldots)$ ***(A1)***

total time is $2 \times (7.081\ldots - 1.919\ldots)$

$10.3258\ldots$

$= 10.3$ (hours) ***A1***

Note: Accept 10.

[3 marks]

(g) A fisherman notes that the water height at nearby Folkestone harbour follows the same sinusoidal pattern as that of Dungeness harbour, with the exception that high tides (and low tides) occur 50 minutes earlier than at Dungeness.

Find a suitable equation that may be used to model the tidal height of water at Folkestone harbour. [2]

Markscheme

METHOD 1

substitutes $t = \frac{11}{3}$ and $H = 6.8$ into their equation for H and attempts to solve for c ***(M1)***

$$6.8 = 2.3\sin\left(\frac{\pi}{6}\left(\frac{11}{3} - c\right)\right) + 4.5 \Rightarrow c = \frac{2}{3}$$

$$H(t) = 2.3\sin\left(\frac{\pi}{6}\left(t - \frac{2}{3}\right)\right) + 4.5$$ ***A1***

METHOD 2

uses their horizontal translation $\left(\frac{12}{4} = 3\right)$ ***(M1)***

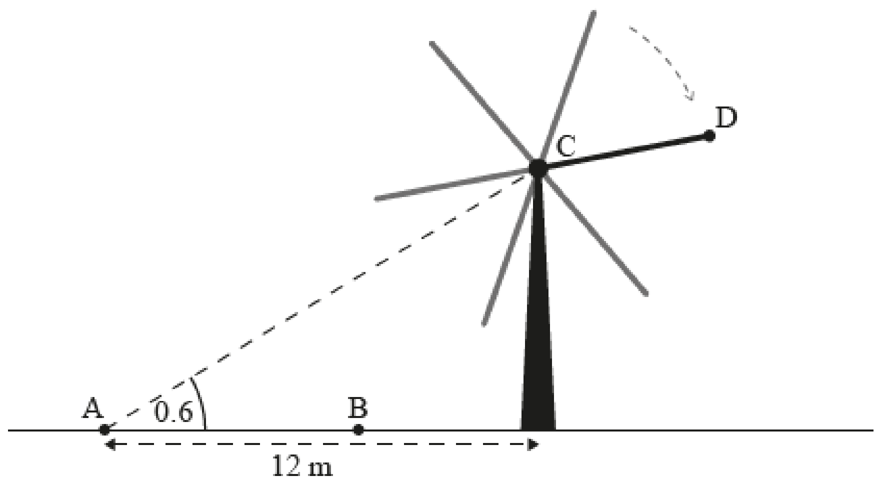
$$\frac{11}{3} - c = 3 \Rightarrow c = \frac{2}{3}$$

$$H(t) = 2.3\sin\left(\frac{\pi}{6}\left(t - \frac{2}{3}\right)\right) + 4.5$$ ***A1***

[2 marks]

3. [Maximum mark: 13]

The six blades of a windmill rotate around a centre point C. Points A and B and the base of the windmill are on level ground, as shown in the following diagram.



From point A the angle of elevation of point C is 0.6 radians.

(a) Given that point A is 12 metres from the base of the windmill, find the height of point C above the ground. [2]

Markscheme

$\tan 0.6 = \frac{h}{12}$ *(M1)*

$8.20964\ldots$

$8.21\,(\mathrm{m})$ *A1*

[2 marks]

An observer walks 7 metres from point A to point B.

(b) Find the angle of elevation of point C from point B. [2]

Markscheme

$\tan B = \frac{8.2096\ldots}{5}$ OR $\tan^{-1} 1.6419\ldots$ *(A1)*

$1.02375\ldots$

1.02 (radians) (accept 58.7°) *A1*

[2 marks]

The observer keeps walking until he is standing directly under point C. The observer has a height of 1.8 metres, and as the blades of the windmill rotate, the end of each blade passes 2.5 metres over his head.

(c) Find the length of each blade of the windmill. [2]

Markscheme

$x+1.8+2.5=8.20964\ldots$ (or equivalent) ***(A1)***

$3.90964\ldots$

$3.91\ (\mathrm{m})$ ***A1***

[2 marks]

One of the blades is painted a different colour than the others. The end of this blade is labelled point D. The height h, in metres, of point D above the ground can be modelled by the function $h(t)=p\cos\left(\frac{3\pi}{10}t\right)+q$, where t is in seconds and $p,\ q\in\mathbb{R}$. When $t=0$, point D is at its maximum height.

(d) Find the value of p and the value of q. [4]

Markscheme

METHOD 1

recognition that blade length = amplitude, $p=\frac{\max-\min}{2}$ ***(M1)***

$p=3.91$ ***A1***

centre of windmill = vertical shift, $q = \frac{\max+\min}{2}$ *(M1)*

$q = 8.21$ ***A1***

METHOD 2

attempting to form two equations in terms of p and q *(M1)(M1)*

$$12.1192\ldots = p\cos\left(\frac{3\pi}{10}\cdot 0\right) + q,\ \ 4.3000\ldots = p\cos\left(\frac{3\pi}{10}\cdot\frac{10}{3}\right) + q$$

$p = 3.91$ ***A1***

$q = 8.21$ ***A1***

[4 marks]

(e) If the observer stands directly under point C for one minute, point D will pass over his head n times.

Find the value of n. [3]

Markscheme

appropriate working towards finding the period *(M1)*

$$\text{period} = \frac{2\pi}{\frac{3\pi}{10}} (= 6.6666\ldots)$$

rotations per minute $= \frac{60}{\text{their period}}$ *(M1)*

$n = 9$ (must be an integer) (accept $n = 10, n = 18, n = 19$) ***A1***

[3 marks]

4. [Maximum mark: 11]

Consider a function f, such that $f(x) = 5.8\sin\left(\frac{\pi}{6}(x+1)\right) + b$, $0 \le x \le 10$, $b \in \mathbb{R}$.

(a) Find the period of f. [2]

Markscheme

correct approach ***A1***

eg $\frac{\pi}{6} = \frac{2\pi}{period}$ (or equivalent)

period = 12 ***A1***

[2 marks]

The function f has a local maximum at the point (2, 21.8) , and a local minimum at (8, 10.2).

(b) Find the value of b. [2]

Markscheme

valid approach ***(M1)***

eg $\frac{\max+\min}{2}$ $b = \max - \text{amplitude}$

$\frac{21.8+10.2}{2}$, or equivalent

$b = 16$ ***A1***

[2 marks]

(c) Hence, find the value of $f(6)$. [2]

Markscheme

attempt to substitute into **their** function ***(M1)***

$5.8\sin\left(\frac{\pi}{6}(6+1)\right)+16$

$f(6) = 13.1$ ***A1***

[2 marks]

A second function g is given by $g(x) = p\sin\left(\frac{2\pi}{9}(x-3.75)\right)+q$, $0 \le x \le 10$; $p, q \in \mathbb{R}$.

The function g passes through the points (3, 2.5) and (6, 15.1).

(d) Find the value of p and the value of q. [5]

Markscheme

valid attempt to set up a system of equations ***(M1)***

two correct equations ***A1***

$p\sin\left(\frac{2\pi}{9}(3-3.75)\right)+q = 2.5$,
$p\sin\left(\frac{2\pi}{9}(6-3.75)\right)+q = 15.1$

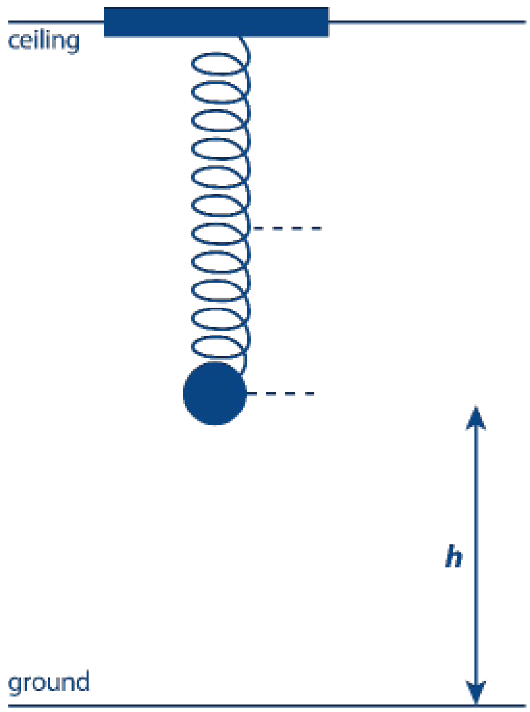
valid attempt to solve system ***(M1)***

$p = 8.4$; $q = 6.7$ ***A1A1***

[5 marks]

5. [Maximum mark: 11]

The following diagram shows a ball attached to the end of a spring, which is suspended from a ceiling.



The height, h metres, of the ball above the ground at time t seconds after being released can be modelled by the function $h(t) = 0.4\cos(\pi t) + 1.8$ where $t \geq 0$.

(a) Find the height of the ball above the ground when it is released. [2]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

attempts to find $h(0)$ **(M1)**

$h(0) = 0.4\cos(0) + 1.8 (= 2.2)$

2.2 (m) (above the ground) **A1**

[2 marks]

(b) Find the minimum height of the ball above the ground. [2]

Markscheme

EITHER

uses the minimum value of $\cos(\pi t)$ which is -1 **M1**

$0.4(-1) + 1.8$ (m)

OR

the amplitude of motion is 0.4 (m) and the mean position is 1.8 (m) **M1**

OR

finds $h\prime(t) = -0.4\pi \sin(\pi t)$, attempts to solve $h\prime(t) = 0$ for t and determines that the minimum height above the ground occurs at $t = 1, 3, \ldots$ **M1**

$0.4(-1) + 1.8$ (m)

THEN

1.4 (m) (above the ground) **A1**

[2 marks]

(c) Show that the ball takes 2 seconds to return to its initial height above the ground for the first time. [2]

Markscheme

EITHER

the ball is released from its maximum height and returns there a period later **R1**

the period is $\frac{2\pi}{\pi}(=2)\ (\text{s})$ **A1**

OR

attempts to solve $h(t)=2.2$ for t **M1**

$\cos(\pi t)=1$

$t=0,\ 2,\ \ldots$ **A1**

THEN

so it takes 2 seconds for the ball to return to its initial position for the first time **AG**

[2 marks]

(d) For the first 2 seconds of its motion, determine the amount of time that the ball is less than $1.8+0.2\sqrt{2}$ metres above the ground. [5]

Markscheme

$0.4\cos(\pi t)+1.8=1.8+0.2\sqrt{2}$ **(M1)**

$0.4\cos(\pi t) = 0.2\sqrt{2}$

$\cos(\pi t) = \frac{\sqrt{2}}{2}$ **A1**

$\pi t = \frac{\pi}{4}, \frac{7\pi}{4}$ **(A1)**

Note: Accept extra correct positive solutions for πt.

$t = \frac{1}{4}, \frac{7}{4}\ (0 \le t \le 2)$ **A1**

Note: Do not award **A1** if solutions outside $0 \le t \le 2$ are also stated.

the ball is less than $1.8 + 0.2\sqrt{2}$ metres above the ground for $\frac{7}{4} - \frac{1}{4}$ (s)

1.5 (s) **A1**

[5 marks]

© International Baccalaureate Organization, 2025