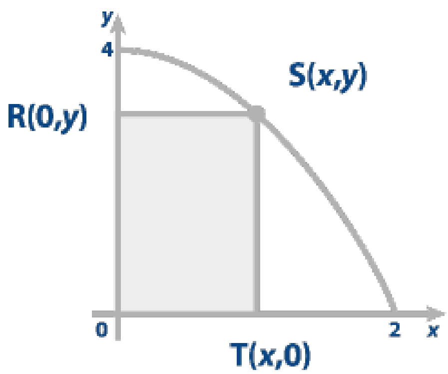
Optimization [63 marks]

1. [Maximum mark: 16]

The following diagram shows the graph of $y = 4 - x^2, 0 \leq x \leq 2$ and rectangle ORST. The rectangle has a vertex at the origin O, a vertex on the y-axis at the point $\mathrm{R}(0,\ y)$, a vertex on the x-axis at the point $\mathrm{T}(x,\ 0)$ and a vertex at point $\mathrm{S}(x,\ y)$ on the graph.



Let P represent the perimeter of rectangle ORST.

(a) Show that $P = -2x^2 + 2x + 8$. [2]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$P = 2x + 2y$ **(A1)**

$= 2x + 2(4 - x^2)$ **A1**

so $P = -2x^2 + 2x + 8$ **AG**

[2 marks]

(b) Find the dimensions of rectangle ORST that has maximum perimeter and determine the value of the maximum perimeter.

[6]

Markscheme

METHOD 1

EITHER

uses the axis of symmetry of a quadratic **(M1)**

$x = -\frac{2}{2(-2)}$

OR

forms $\frac{\mathrm{d}P}{\mathrm{d}x} = 0$ **(M1)**

$-4x + 2 = 0$

THEN

$x = \frac{1}{2}$ **A1**

substitutes their value of x into $y = 4 - x^2$ **(M1)**

$y = 4 - \left(\frac{1}{2}\right)^2$

$y = \frac{15}{4}$ **A1**

so the dimensions of rectangle ORST of maximum perimeter are $\frac{1}{2}$ by $\frac{15}{4}$

EITHER

substitutes their value of x into $P = -2x^2 + 2x + 8$ **(M1)**

$P = -2\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) + 8$

OR

substitutes their values of x and y into $P = 2x + 2y$ **(M1)**

$P = 2\left(\frac{1}{2}\right) + 2\left(\frac{15}{4}\right)$

$P = \frac{17}{2}$ **A1**

so the maximum perimeter is $\frac{17}{2}$

METHOD 2

attempts to complete the square **M1**

$P = -2\left(x - \frac{1}{2}\right)^2 + \frac{17}{2}$ **A1**

$x = \frac{1}{2}$ **A1**

substitutes their value of x into $y = 4 - x^2$ **(M1)**

$y = 4 - \left(\frac{1}{2}\right)^2$

$y = \frac{15}{4}$ **A1**

so the dimensions of rectangle ORST of maximum perimeter are $\frac{1}{2}$ by $\frac{15}{4}$

$P = \frac{17}{2}$ **A1**

so the maximum perimeter is $\frac{17}{2}$

[6 marks]

Let A represent the area of rectangle ORST.

(c) Find an expression for A in terms of x. [2]

Markscheme

substitutes $y = 4 - x^2$ into $A = xy$ **(M1)**

$A = x(4 - x^2)\ \ (= 4x - x^3)$ **A1**

[2 marks]

(d) Find the dimensions of rectangle ORST that has maximum area. [5]

Markscheme

$\frac{\mathrm{d}A}{\mathrm{d}x} = 4 - 3x^2$ **A1**

attempts to solve their $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ for x **(M1)**

$4 - 3x^2 = 0$

$\Rightarrow x = \frac{2}{\sqrt{3}} \left(= \frac{2\sqrt{3}}{3}\right)\ (x > 0)$ **A1**

substitutes their (positive) value of x into $y = 4 - x^2$ **(M1)**

$y = 4 - \left(\frac{2}{\sqrt{3}}\right)^2$

$y = \frac{8}{3}$ **A1**

[5 marks]

(e) Determine the maximum area of rectangle ORST. [1]

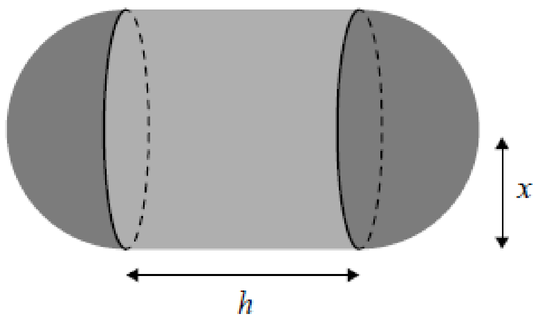
Markscheme

$A = \frac{16}{3\sqrt{3}} \left(= \frac{16\sqrt{3}}{9}\right)$ **A1**

[1 mark]

2. [Maximum mark: 14]

The solid shown in the following diagram is comprised of a cylinder and two hemispheres. The cylinder has height h cm and radius x cm. The hemispheres fit exactly onto either end of the cylinder.



The volume of the cylinder is 41 cm^3.

(a) Show that the total surface area, $S \text{ cm}^2$, of the solid is given by $S = \frac{82}{x} + 4\pi x^2$. [3]

Markscheme

$\pi x^2 h = 41$ ***(A1)***

attempt to rearrange AND substitute their h into the expression for the total surface area ***(M1)***

$S = 2\pi x\left(\frac{41}{\pi x^2}\right) + 4\pi x^2$ ***A1***

$S = \frac{82}{x} + 4\pi x^2$ ***AG***

[3 marks]

The total surface area of the solid has a local maximum or a local minimum value when $x = a$.

(b.i) Find an expression for $\frac{dS}{dx}$. [2]

Markscheme

$\frac{\mathrm{d}S}{\mathrm{d}x} = -\frac{82}{x^2} + 8\pi x$ (or equivalent) ***A1A1***

Note: Award ***A1*** for each correct term.

Award ***A1A0*** if additional terms are given.

[2 marks]

(b.ii) Hence, find the **exact** value of a. [3]

Markscheme

$\frac{\mathrm{d}S}{\mathrm{d}x} = 0$ ***(M1)***

$-\frac{82}{a^2} + 8\pi a = 0$ ***(A1)***

$(a =)\left(\frac{82}{8\pi}\right)^{\frac{1}{3}}$ ***A1***

[3 marks]

(c.i) Find an expression for $\frac{\mathrm{d}^2S}{\mathrm{d}x^2}$. [2]

Markscheme

$\frac{\mathrm{d}^2S}{\mathrm{d}x^2} = 164x^{-3} + 8\pi$ (or equivalent) ***A1A1***

Note: Award ***A1*** for each correct term.

Award ***A1A0*** if additional terms are given.

[2 marks]

(c.ii) Use the second derivative of S to justify that S is a minimum when $x = a$. [2]

Markscheme

EITHER

substituting their value of x into their $\frac{\mathrm{d}^2 S}{\mathrm{d}x^2}$ ***(M1)***

$\frac{\mathrm{d}^2 S}{\mathrm{d}x^2} = 75.3982\ldots$

$= 75.4\ (= 24\pi) > 0 (75.7 \text{ from } a = 1.48)$ ***A1***

OR

sketch of the graph of $\frac{\mathrm{d}^2 S}{\mathrm{d}x^2}$ with their value of x clearly indicated ***(M1)***

$\frac{\mathrm{d}^2 S}{\mathrm{d}x^2} > 0$ at $x = a$ ***A1***

THEN

therefore S is a minimum ***A1***

[2 marks]

(c.iii) Find the minimum surface area of the solid. [2]

Markscheme

attempt to substitute their value of a into S OR use of graph of S ***(M1)***

$82.9304\ldots$

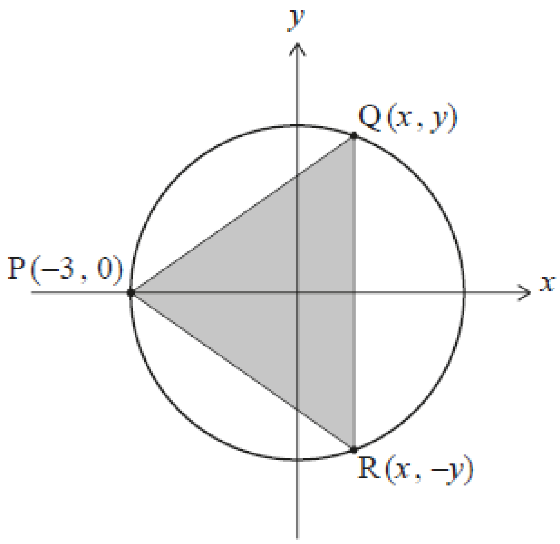
minimum surface area = $82.9\ (\mathrm{cm}^2)$ ***A1***

[2 marks]

3. [Maximum mark: 14]

A circle with equation $x^2 + y^2 = 9$ has centre $(0,\ 0)$ and radius 3.

A triangle, PQR, is inscribed in the circle with its vertices at $\mathrm{P}(-3,\ 0)$, $\mathrm{Q}(x,\ y)$ and $\mathrm{R}(x,\ -y)$, where Q and R are variable points in the first and fourth quadrants respectively. This is shown in the following diagram.



(a) For point Q, show that $y = \sqrt{9 - x^2}$. [1]

Markscheme

$y^2 = 9 - x^2$ OR $y = \pm\sqrt{9 - x^2}$ ***A1***

(since $y > 0$) $\Rightarrow y = \sqrt{9 - x^2}$ ***AG***

[1 mark]

(b) Hence, find an expression for A, the area of triangle PQR, in terms of x. [3]

Markscheme

$b = 2y \left(= 2\sqrt{9 - x^2}\right)$ or $h = x + 3$ ***(A1)***

attempts to substitute their base expression and height expression into $A = \frac{1}{2}bh$ ***(M1)***

$A = \sqrt{9 - x^2}\left(x + 3\right)$ (or equivalent) $\left(= \frac{2\left(x+3\right)\sqrt{9-x^2}}{2} = x\sqrt{9 - x^2} + 3\sqrt{9 - x^2}\right)$ ***A1***

[3 marks]

(c) Show that $\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{9-3x-2x^2}{\sqrt{9-x^2}}$. [4]

Markscheme

METHOD 1

attempts to use the product rule to find $\frac{\mathrm{d}A}{\mathrm{d}x}$ ***(M1)***

attempts to use the chain rule to find $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{9-x^2}$ ***(M1)***

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} =\right)\sqrt{9-x^2} + \left(3+x\right)\left(\frac{1}{2}\right)\left(9-x^2\right)^{-\frac{1}{2}}\left(-2x\right)\left(= \sqrt{9-x^2} - \frac{x^2+3x}{\sqrt{9-x^2}}\right)$

A1

$\left(\frac{\mathrm{d}A}{\mathrm{d}x} =\right)\frac{9-x^2}{\sqrt{9-x^2}} - \frac{x^2+3x}{\sqrt{9-x^2}} \left(= \frac{9-x^2-(x^2+3x)}{\sqrt{9-x^2}}\right)$ ***A1***

$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{9-3x-2x^2}{\sqrt{9-x^2}}$ ***AG***

METHOD 2

$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\mathrm{d}A}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x}$

attempts to find $\frac{\mathrm{d}A}{\mathrm{d}y}$ where $A = y(x+3)$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$ where $y^2 = 9 - x^2$ ***(M1)***

$\frac{\mathrm{d}A}{\mathrm{d}y} = y\frac{\mathrm{d}x}{\mathrm{d}y} + x + 3$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ (or equivalent) ***A1***

substitutes their $\frac{\mathrm{d}A}{\mathrm{d}y}$ and their $\frac{\mathrm{d}y}{\mathrm{d}x}$ into $\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\mathrm{d}A}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}A}{\mathrm{d}x} = \left(y\left(-\frac{y}{x}\right) + x + 3\right)\left(-\frac{x}{y}\right)$ (or equivalent)

$= \frac{9-x^2-x^2-3x}{\sqrt{9-x^2}}$ (or equivalent) ***A1***

$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{9-3x-2x^2}{\sqrt{9-x^2}}$ ***AG***

[4 marks]

(d) Hence or otherwise, find the y-coordinate of R such that A is a maximum. [6]

Markscheme

$\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \left(\frac{9-3x-2x^2}{\sqrt{9-x^2}} = 0\right)$ ***(M1)***

attempts to solve $9 - 3x - 2x^2 = 0$ (or equivalent) ***(M1)***

$-(2x-3)(x+3)(=0)$ OR $x = \frac{3 \pm \sqrt{(-3)^2 - 4(-2)(9)}}{2(-2)}$ (or equivalent) ***(A1)***

$x = \frac{3}{2}$ ***A1***

Note: Award the above ***A1*** if $x = -3$ is also given.

substitutes their value of x into either $y = \sqrt{9-x^2}$ or $y = -\sqrt{9-x^2}$

Note: Do not award the above ***(M1)*** if $x \leq 0$. ***(M1)***

$y = -\sqrt{9 - \left(\frac{3}{2}\right)^2}$

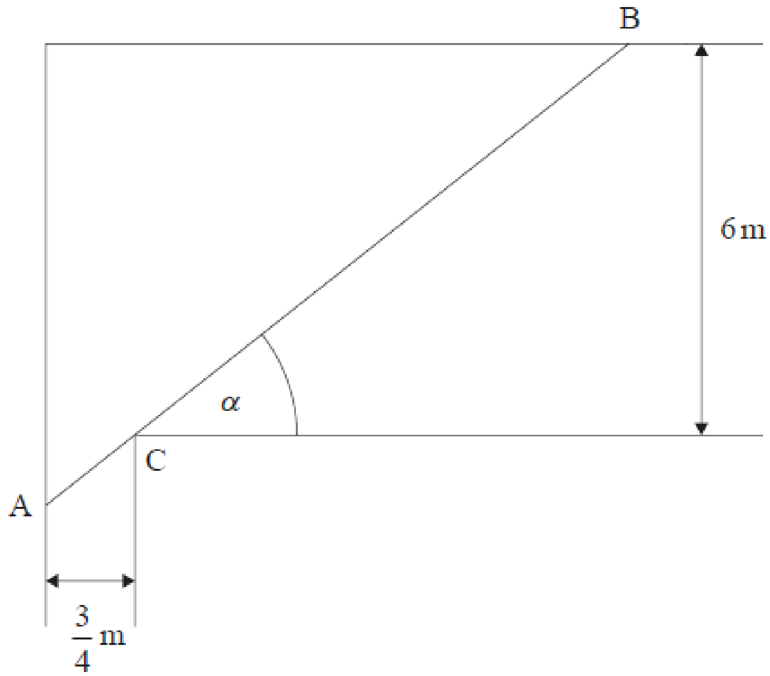
$= -\frac{\sqrt{27}}{2} \left(= -\frac{3\sqrt{3}}{2}, = -\sqrt{\frac{27}{4}}, = -\sqrt{6.75}\right)$ ***A1***

[6 marks]

4. [Maximum mark: 19]

Consider the following diagram, which shows the plan of part of a house.

diagram not to scale



A narrow passageway with $\frac{3}{4}$ m width is perpendicular to a room of width 6 m. There is a corner at point C. Points A and B are variable points on the base of the walls such that A, C and B lie on a straight line.

Let L denote the length AB in metres.

Let α be the angle that $[\text{AB}]$ makes with the room wall, where $0 < \alpha < \frac{\pi}{2}$.

(a) Show that $L = \frac{3}{4}\sec\,\alpha + 6\text{ cosec }\alpha$. [2]

Markscheme

$L = \text{AC} + \text{CB}$

$\frac{\left(\frac{3}{4}\right)}{\text{AC}} = \cos\,\alpha \left(\Rightarrow \text{AC} = \frac{\frac{3}{4}}{\cos\,\alpha} \Rightarrow \text{AC} = \frac{3}{4}\sec\,\alpha\right)$ ***A1***

$\frac{6}{\text{CB}} = \sin\,\alpha \left(\Rightarrow \text{CB} = \frac{6}{\sin\,\alpha} \Rightarrow \text{CB} = 6\,\text{cosec}\,\alpha\right)$ ***A1***

so $L = \frac{3}{4}\sec\,\alpha + 6\,\text{cosec}\,\alpha$ ***AG***

[2 marks]

(b.i) Find $\frac{\mathrm{d}L}{\mathrm{d}\alpha}$. [1]

Markscheme

$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \frac{3}{4}\sec\,\alpha\,\tan\,\alpha - 6\,\text{cosec}\,\alpha\,\cot\,\alpha$ ***A1***

[1 mark]

(b.ii) When $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, show that $\alpha = \arctan\,2$. [4]

Markscheme

attempt to write $\frac{\mathrm{d}L}{\mathrm{d}\alpha}$ in terms of $\sin\,\alpha$, $\cos\,\alpha$ or $\tan\,\alpha$ (may be seen in (i)) ***(M1)***

$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \frac{\frac{3}{4}\sin\,\alpha}{\cos^2\,\alpha} - \frac{6\,\cos\,\alpha}{\sin^2\,\alpha}$ OR $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = \frac{\frac{3}{4}\tan\,\alpha}{\cos\,\alpha} - \frac{6}{\sin\,\alpha\,\cos\,\alpha}\left(= \frac{\frac{3}{4}\tan^3\,\alpha - 6}{\cos\,\alpha\,\tan^2\,\alpha}\right)$

$\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0 \Rightarrow \frac{3}{4}\sin^3\,\alpha - 6\cos^3\,\alpha = 0$ OR $\frac{3}{4}\tan^3\,\alpha - 6 = 0$ (or equivalent) ***(A1)***

$\tan^3\,\alpha = 8$ ***A1***

$\tan\,\alpha = 2$ ***A1***

$\alpha = \arctan\,2$ ***AG***

[4 marks]

(c.i) Find $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}$.

[3]

Markscheme

attempt to use product rule (at least once) ***(M1)***

$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{3}{4}\sec\,\alpha\,\tan\,\alpha\,\tan\,\alpha + \frac{3}{4}\sec\,\alpha\,\sec^2\,\alpha$

$+6\,\text{cosec}\,\alpha\,\cot\,\alpha\,\cot\,\alpha + 6\,\text{cosec}\,\alpha\,\text{cosec}^2\,\alpha$ ***A1A1***

Note: Award ***A1*** for $\frac{3}{4}\sec\,\alpha\,\tan\,\alpha\,\tan\,\alpha + \frac{3}{4}\sec\,\alpha\,\sec^2\,\alpha$ and ***A1*** for $+6\,\text{cosec}\,\alpha\,\cot\,\alpha\,\cot\,\alpha + 6\,\text{cosec}\,\alpha\,\text{cosec}^2\,\alpha$.

Allow unsimplified correct answer.

$\left(\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{3}{4}\sec\,\alpha\,\tan^2\,\alpha + \frac{3}{4}\sec^3\,\alpha + 6\,\text{cosec}\,\alpha\,\cot^2\,\alpha + 6\,\text{cosec}^3\,\alpha\right)$

[3 marks]

(c.ii) When $\alpha = \arctan\,2$, show that $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{45}{4}\sqrt{5}$. [4]

Markscheme

attempt to find a ratio other than $\tan\,\alpha$ using an appropriate trigonometric identity OR a right triangle with at least two side lengths seen ***(M1)***

Note: Award ***M0*** for $\alpha = \arctan\,2$ substituted into their $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}$ with no further progress.

one correct ratio ***(A1)***

$\sec\,\alpha = \sqrt{5}$ OR $\operatorname{cosec}\,\alpha = \frac{\sqrt{5}}{2}$ OR $\cot\,\alpha = \frac{1}{2}$ OR $\cos\,\alpha = \frac{1}{\sqrt{5}}$ OR $\sin\,\alpha = \frac{2}{\sqrt{5}}$

Note: ***M1A1*** may be seen in part (d).

$\frac{3}{4}\left(\sqrt{5}\right)\left(2^2\right) + \frac{3}{4}\left(\sqrt{5}\right)^3 + 6\left(\frac{\sqrt{5}}{2}\right)\left(\frac{1}{2}\right)^2 + 6\left(\frac{\sqrt{5}}{2}\right)^3$ (or equivalent)
A2

$\frac{12\sqrt{5}}{4} + \frac{15\sqrt{5}}{4} + \frac{3\sqrt{5}}{4} + \frac{15\sqrt{5}}{4}$

Note: Award ***A1*** for only two or three correct terms.

Award a maximum of ***(M1)(A1)A1*** on ***FT*** from c(i).

$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}\,\frac{45}{4}\sqrt{5}$ ***AG***

[4 marks]

(d.i) Hence, justify that L is a minimum when $\alpha = \arctan 2$. [1]

Markscheme

$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} > 0$ OR concave up (or equivalent) ***R1***

(and $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, when $\alpha = \arctan 2$, hence L is a minimum)

[1 mark]

(d.ii) Determine this minimum value of L. [2]

Markscheme

$(L_{\min} =) \frac{3}{4}\left(\sqrt{5}\right) + 6\left(\frac{\sqrt{5}}{2}\right)$ ***(A1)***

$= \frac{15\sqrt{5}}{4}$ ***A1***

[3 marks]

Two people need to carry a pole of length $11.25\,\mathrm{m}$ from the passageway into the room. It must be carried horizontally.

(e) Determine whether this is possible, giving a reason for your answer. [2]

Markscheme

$\left(11.25 =\right) \frac{15\sqrt{9}}{4} > \frac{15\sqrt{5}}{4}$ (or equivalent comparative reasoning) ***R1***

the pole cannot be carried (horizontally from the passageway into the room) ***A1***

Note: Do not award ***R0A1.***

[2 marks]