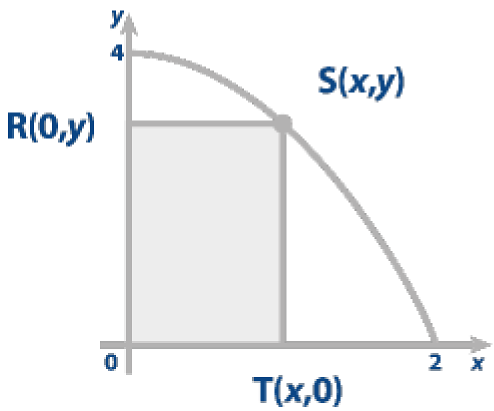
Optimization [63 marks]

1. [Maximum mark: 16]

The following diagram shows the graph of $y = 4 - x^2, 0 \leq x \leq 2$ and rectangle ORST. The rectangle has a vertex at the origin O, a vertex on the y-axis at the point $\mathrm{R}(0,\ y)$, a vertex on the x-axis at the point $\mathrm{T}(x,\ 0)$ and a vertex at point $\mathrm{S}(x,\ y)$ on the graph.



Let P represent the perimeter of rectangle ORST.

(a) Show that $P = -2x^2 + 2x + 8$. [2]

(b) Find the dimensions of rectangle ORST that has maximum perimeter and determine the value of the maximum perimeter. [6]

Let A represent the area of rectangle ORST.

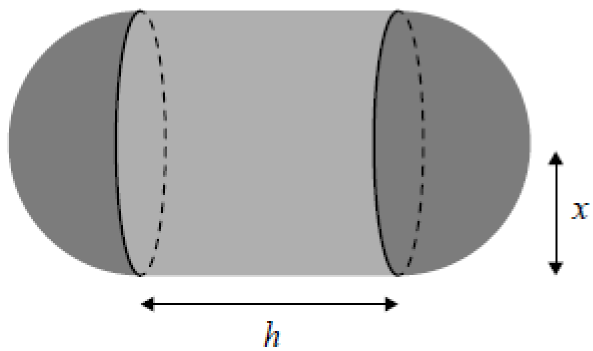
(c) Find an expression for A in terms of x. [2]

(d) Find the dimensions of rectangle ORST that has maximum area. [5]

(e) Determine the maximum area of rectangle ORST. [1]

2. [Maximum mark: 14]

The solid shown in the following diagram is comprised of a cylinder and two hemispheres. The cylinder has height h cm and radius x cm. The hemispheres fit exactly onto either end of the cylinder.



The volume of the cylinder is $41\ \text{cm}^3$.

(a) Show that the total surface area, $S\ \text{cm}^2$, of the solid is given by $S = \frac{82}{x} + 4\pi x^2$. [3]

The total surface area of the solid has a local maximum or a local minimum value when $x = a$.

(b.i) Find an expression for $\frac{\mathrm{d}S}{\mathrm{d}x}$. [2]

(b.ii) Hence, find the **exact** value of a. [3]

(c.i) Find an expression for $\frac{\mathrm{d}^2S}{\mathrm{d}x^2}$. [2]

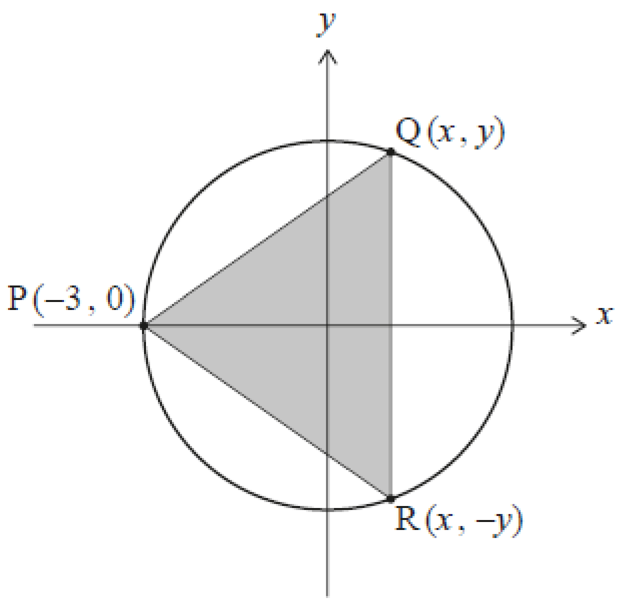
(c.ii) Use the second derivative of S to justify that S is a minimum when $x = a$. [2]

(c.iii) Find the minimum surface area of the solid. [2]

3. [Maximum mark: 14]

A circle with equation $x^2 + y^2 = 9$ has centre $(0,\ 0)$ and radius 3.

A triangle, PQR, is inscribed in the circle with its vertices at $\mathrm{P}(-3,\ 0)$, $\mathrm{Q}(x,\ y)$ and $\mathrm{R}(x,\ -y)$, where Q and R are variable points in the first and fourth quadrants respectively. This is shown in the following diagram.



(a) For point Q, show that $y = \sqrt{9 - x^2}$. [1]

(b) Hence, find an expression for A, the area of triangle PQR, in terms of x. [3]

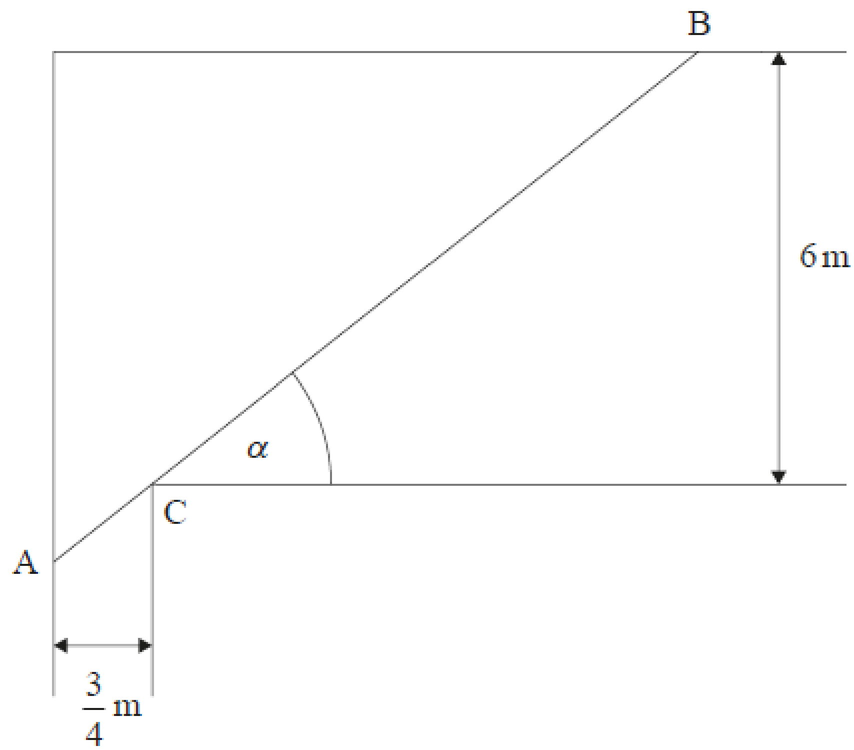
(c) Show that $\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{9-3x-2x^2}{\sqrt{9-x^2}}$. [4]

(d) Hence or otherwise, find the y-coordinate of R such that A is a maximum. [6]

4. [Maximum mark: 19]

Consider the following diagram, which shows the plan of part of a house.

diagram not to scale



A narrow passageway with $\frac{3}{4}$ m width is perpendicular to a room of width 6 m. There is a corner at point C. Points A and B are variable points on the base of the walls such that A, C and B lie on a straight line.

Let L denote the length AB in metres.

Let α be the angle that $[\mathrm{AB}]$ makes with the room wall, where $0 < \alpha < \frac{\pi}{2}$.

(a) Show that $L = \frac{3}{4}\sec \alpha + 6 \operatorname{cosec} \alpha$. [2]

(b.i) Find $\frac{\mathrm{d}L}{\mathrm{d}\alpha}$. [1]

(b.ii) When $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, show that $\alpha = \arctan 2$. [4]

(c.i) Find $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}$. [3]

(c.ii) When $\alpha = \arctan 2$, show that $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{45}{4}\sqrt{5}$. [4]

(d.i) Hence, justify that L is a minimum when $\alpha = \arctan 2$. [1]

(d.ii) Determine this minimum value of L. [2]

Two people need to carry a pole of length $11.25\,\mathrm{m}$ from the passageway into the room. It must be carried horizontally.

(e) Determine whether this is possible, giving a reason for your answer. [2]