l'Hopital [46 marks]

1. [Maximum mark: 5] EXN.1.AHL.TZ0.6

Use l'Hôpital's rule to determine the value of $\lim_{x \to 0} \left(\frac{2x \cos(x^2)}{5 \tan x} \right)$. [5]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

attempts to apply l'Hôpital's rule on $\lim_{x \to 0} \left(\frac{2x \cos(x^2)}{5 \tan x} \right)$ **M1**

$= \lim_{x \to 0} \left(\frac{2 \cos(x^2) - 4x^2 \sin(x^2)}{5 \sec^2 x} \right)$ **M1A1A1**

Note: Award **M1** for attempting to use product and chain rule differentiation on the numerator, **A1** for a correct numerator and **A1** for a correct denominator. The awarding of **A1** for the denominator is independent of the **M1**.

$= \frac{2}{5}$ **A1**

[5 marks]

2. [Maximum mark: 6] 24M.1.AHL.TZ2.8

Use l'Hôpital's rule to find $\lim_{x \to 0} \frac{\sec^4 x - \cos^2 x}{x^4 - x^2}$. [6]

Markscheme

Note: To award full marks limit notation $\lim\limits_{x\to 0}$ must be seen at least once in their working. If no limit notation is seen but otherwise all correct, do not award the final ***A1***.

$\lim\limits_{x\to 0}\frac{4\sec^2 x\tan x+2\sin x\cos x}{4x^3-2x}$ ***A1A1***

Note: Award ***A1*** for numerator and ***A1*** for denominator.

$=\lim\limits_{x\to 0}\frac{16 sec^4 x\tan^2 x+4 sec^6 x-2\sin^2 x+2\cos^2 x}{12x^2-2}$ ***M1A1A1***

Note: Award ***M1 for second use of*** l'Hôpital's rule providing their expression is in indeterminate form as $x\to 0$ and providing there is **no third attempt** at using l'Hôpital's Rule.

$=-3$ ***A1***

[6 marks]

3. [Maximum mark: 8] 22M.2.AHL.TZ2.7

Consider $\lim\limits_{x\to 0}\frac{\arctan(\cos x)-k}{x^2}$, where $k\in\mathbb{R}$.

(a) Show that a finite limit only exists for $k=\frac{\pi}{4}$. [2]

Markscheme

(as $\lim\limits_{x\to 0}x^2=0$, the indeterminate form $\frac{0}{0}$ is required for the limit to exist)

$\Rightarrow \lim_{x \to 0}(\arctan(\cos x) - k) = 0$ ***M1***

$\arctan 1 - k = 0 \ \ (k = \arctan 1)$ ***A1***

so $k = \frac{\pi}{4}$ ***AG***

Note: Award ***M1A0*** for using $k = \frac{\pi}{4}$ to show the limit is $\frac{0}{0}$.

[2 marks]

(b) Using l'Hôpital's rule, show algebraically that the value of the limit is $-\frac{1}{4}$. [6]

Markscheme

$\lim_{x \to 0} \frac{\arctan(\cos x) - \frac{\pi}{4}}{x^2} \left(= \frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{\frac{-\sin x}{1+\cos^2 x}}{2x}$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

recognises to apply l'Hôpital's rule again ***(M1)***

$= \lim_{x \to 0} \frac{\frac{-\sin x}{1+\cos^2 x}}{2x} \ \left(= \frac{0}{0}\right)$

Note: Award ***M0*** if their limit is not the indeterminate form $\frac{0}{0}$.

EITHER

$$= \lim_{x \to 0} \frac{\frac{-\cos x\left(1+\cos^2 x\right)-2\sin^2 x\cos x}{\left(1+\cos^2 x\right)^2}}{2}$$ ***A1A1***

Note: Award ***A1*** for a correct first term in the numerator and ***A1*** for a correct second term in the numerator.

OR

$$\lim_{x \to 0} \frac{-\cos x}{2(1+\cos^2 x)-4x\sin x\cos x}$$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

THEN

substitutes $x = 0$ into the correct expression to evaluate the limit ***A1***

Note: The final ***A1*** is dependent on all previous marks.

$$= -\frac{1}{4}$$ ***AG***

[6 marks]

4. [Maximum mark: 5] 21M.1.AHL.TZ1.8

Use l'Hôpital's rule to find $\lim_{x \to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$. [5]

Markscheme

attempt to differentiate numerator and denominator ***M1***

$$\lim_{x\to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$$

$$=\lim_{x\to 0}\frac{\left(\frac{2}{1+4x^2}\right)}{3\sec^2 3x}$$ ***A1A1***

Note: ***A1*** for numerator and ***A1*** for denominator. Do not condone absence of limits.

attempt to substitute $x=0$ ***(M1)***

$$=\frac{2}{3}$$ ***A1***

Note: Award a maximum of ***M1A1A0M1A1*** for absence of limits.

[5 marks]

5. [Maximum mark: 6] 20N.1.AHL.TZ0.F_1

Use l'Hôpital's rule to determine the value of

$$\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}.$$ [6]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

using l'Hôpital's rule,

$$\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}=\lim_{x\to 0}\frac{2\cos x-2\cos 2x}{3x^2}$$ ***M1A1***

$$=\lim_{x\to 0}\frac{-2\sin x+4\sin 2x}{6x}$$ ***(M1)A1***

$= \lim_{x \to 0} \frac{-2 \cos x + 8 \cos 2x}{6}$ ***A1***

$= 1$ ***A1***

[6 marks]

6. [Maximum mark: 7] 20N.3.AHL.TZ0.Hca_1

Use l'Hôpital's rule to find

$\lim_{x \to 1} \frac{\cos(x^2 - 1) - 1}{e^{x-1} - x}$. [7]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to use l'Hôpital's rule ***M1***

$= \lim_{x \to 1} \frac{-2x \sin(x^2 - 1)}{e^{x-1} - 1}$ ***A1A1***

Note: Award ***A1*** for the numerator and ***A1*** for the denominator.

substitution of 1 into their expression ***(M1)***

$= \frac{0}{0}$ hence use l'Hôpital's rule again

Note: If the first use of l'Hôpital's rule results in an expression which is not in indeterminate form, do not award any further marks.

attempt to use product rule in numerator ***M1***

$= \lim_{x \to 1} \frac{-4x^2 \cos(x^2 - 1) - 2 \sin(x^2 - 1)}{e^{x-1}}$ ***A1***

$= -4$ ***A1***

[7 marks]

7. [Maximum mark: 9] 19M.3.AHL.TZ0.Hca_4

Using L'Hôpital's rule, find $\lim_{x \to 0} \left(\frac{\tan 3x - 3\tan x}{\sin 3x - 3\sin x} \right)$. [9]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\lim_{x \to 0} \left(\frac{\tan 3x - 3\tan x}{\sin 3x - 3\sin x} \right)$

$\lim_{x \to 0} \left(\frac{3\sec^2 3x - 3\sec^2 x}{3\cos 3x - 3\cos x} \right) \quad \left(= \lim_{x \to 0} \left(\frac{\sec^2 3x - \sec^2 x}{\cos 3x - \cos x} \right) \right)$ ***M1A1A1***

Note: Award ***M1*** for attempt at differentiation using l'Hopital's rule, ***A1*** for numerator, ***A1*** for denominator.

METHOD 1

using l'Hopital's rule again

$= \lim_{x \to 0} \left(\frac{18\sec^2 3x \tan 3x - 6\sec^2 x \tan x}{-9\sin 3x + 3\sin x} \right) \quad \left(= \lim_{x \to 0} \left(\frac{6\sec^2 3x \tan 3x - 2\sec^2 x \tan x}{-3\sin 3x + \sin x} \right) \right)$ ***A1A1***

EITHER

$= \lim_{x \to 0} \left(\frac{108\sec^2 3x \tan^2 3x + 54\sec^4 3x - 12\sec^2 x \tan^2 x - 6\sec^4 x}{\text{-}\ 27\cos 3x + 3\cos x} \right)$ ***A1A1***

Note: Not all terms in numerator need to be written in final fraction. Award ***A1*** for $54\sec^4 3x + \ldots - 6\sec^4 x \ldots -$. However, if the terms are written, they must be correct to award A1.

attempt to substitute $x = 0$ ***M1***

$= \frac{48}{-24}$

OR

$\frac{\mathrm{d}}{\mathrm{d}x}\left(18\sec^2 3x \tan 3x - 6\sec^2 x \tan x\right)|_{x=0} = 48$ ***(M1)A1***

$\frac{\mathrm{d}}{\mathrm{d}x}(-9\sin 3x + 3\sin x)|_{x=0} = -24$ ***A1***

THEN

$\left(\lim_{x\to 0}\left(\frac{\tan 3x - 3\tan x}{\sin 3x - 3\sin x}\right)\right) = -2$ ***A1***

METHOD 2

$= \lim_{x\to 0}\left(\frac{\frac{3}{\cos^2 3x} - \frac{3}{\cos^2 x}}{3\cos 3x - 3\cos x}\right)$ ***M1***

$= \lim_{x\to 0}\left(\frac{\cos^2 x - \cos^2 3x}{\cos^2 3x \cos^2 x(\cos 3x - \cos x)}\right)$ ***A1***

$= \lim_{x\to 0}\left(\frac{\cos x + \cos 3x}{-\cos^2 3x \cos^2 x}\right)$ ***M1A1***

attempt to substitute $x = 0$ ***M1***

$= \frac{2}{-1}$

$= -2$ ***A1***

[9 marks]