Name:

Mathematics AA HL

January 31, 2025

1 hour 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Calculators are **not allowed** for this paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly.
- The maximum mark for this examination paper is [**72 marks**].
- Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to **show all working**.
- Write your solutions to section A in the space provided and solutions to section B on separate sheets.

SECTION A

1. [4 *points*]

Show from **first principles** that the derivative of $y = \sqrt{x} + x$ is given by $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + 1$.

2. [4 *points*]

Solve the equation:

$$\cos 4x = \sin 2x$$

for $0 \leqslant x < 2\pi$.

3. [6 *points*]

Values of $f(x)$, $f'(x)$, $g(x)$ and $g'(x)$ for some values of x are displayed in the table below:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	2	1	4	3
2	3	-2	7	-5
3	1.5	2	-0.5	1
4	3	5	3	1

Let $p(x) = f(g(x))$ and $q(x) = f(x)g(x)$. Find $p'(1)$ and $q'(p(1))$.

5. [6 *points*]

Bolzano's Theorem states that if a continuous function is defined on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ has a zero in the interval $[a, b]$.

Consider the function $f(x) = x^3 + x^2 + 2x + 1$.

(a) Show that $f(x) = 0$ for some $x \in [-1, 0]$. [3 *points*]

(b) Show that $f(x)$ is increasing for all x, and hence deduce the number of real solutions to the equation $f(x) = 0$. Justify your answer. [3 *points*]

6. [5 *points*]

Let $f(x) = \sqrt{\pi - 6 \arccos x}$.

(a) Find the domain and range of $f(x)$.

(b) Find $f^{-1}(x)$, the inverse of $f(x)$, and state its domain and range.

7. [5 *points*]

Find the area of the triangle enclosed by the tangent to the curve $y = \dfrac{\arctan x}{x}$ at $x = 1$ and the axes.

8. [5 *points*]

Solve the equation:

$$\log_3(x^2) = \log_x(9x^3)$$

SECTION B

9. [16 *points*]

Let $f(x) = x\sqrt{9-x^2} + 2\arcsin(\frac{x}{3})$.

a) Find the largest possible domain D for the function f. [2 *points*]

For the following parts it is assumed that $x \in D$.

b) Find $f'(x)$ in simplified form and hence find the x-coordinate of the maximum point of the graph of $y = f(x)$. Justify that it is a maximum. [7 *points*]

c) Show that $f''(x) = \dfrac{x(2x^2-25)}{(9-x^2)\sqrt{9-x^2}}$ and hence find coordinates of any points of inflexion of the graph of $y = f(x)$. Justify your answer. [7 *points*]

10. [21 *points*]

(a) Consider the function $f(x) = \ln x - x$. [6 *points*]

(i) Find the coordinates of the stationary point on the graph of $f(x)$.

(ii) Show that the point is a maximum.

(iii) Deduce that

$$\ln x \leqslant x - 1$$

for $x > 0$ with equality only when $x = 1$.

(b) By considering the stationary point of $g(x) = \ln x + \frac{1}{x}$. show that

$$\ln x \geqslant \frac{x-1}{x}$$

for $x > 0$ with equality only when $x = 1$. [6 *points*]

(c) By putting $x = \dfrac{z}{y}$ where $0 < y < z$ into the inequality from part (b), deduce Napier's inequality:

[3 *points*]

$$\frac{1}{z} < \frac{\ln z - \ln y}{z - y} < \frac{1}{y}$$

(d) Use Napier's inequality with $y = n$ and $z = n + 1$ to show that: [2 *points*]

$$\frac{n}{n+1} < n \ln \frac{n+1}{n} < 1$$

(e) Using the inequality from part (d) and the Squeeze Theorem show that: [4 *points*]

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$