43 Show if a is a constant that

a) $\dfrac{d}{dx}\left[\arctan\left(\dfrac{x}{a}\right)\right] = \dfrac{a}{a^2 + x^2}$ b) $\dfrac{d}{dx}\left[\arcsin\left(\dfrac{x}{a}\right)\right] = \dfrac{1}{\sqrt{a^2 - x^2}}$

44 Find the equation of the line tangent to the curve $y = 4x \arctan 2x$ at the point on the curve where $x = \frac{1}{2}$. Express the equation exactly in the form $y = mx + c$, where m and c are constants.

45 Consider the function $f(x) = \arcsin(\cos x)$ with domain of $0 \leqslant x < \pi$.

a) Prove that f is a linear function.

b) Express the function exactly in the form $f(x) = ax + b$, where a and b are constants.

46 A 3-metre tall statue is on top of a column such that the bottom of the statue is 2 metres above the eye level of a person viewing the statue. How far from the base of the column should the person stand to get the best view of the statue, that is, so that the angle subtended at the observer's eye by the statue is a maximum?

47 A particle moves along the x-axis so that its displacement, s (in metres), from the origin at any time $t \geqslant 0$ (in seconds) is given by $s(t) = \arctan\sqrt{t}$.

a) Find the exact velocity of the particle at (i) $t = 1$ second, and at (ii) $t = 4$ seconds.

b) Find the exact acceleration of the particle at (i) $t = 1$ second, and at (ii) $t = 4$ seconds.

c) Describe the motion of the particle.

d) What is the limiting displacement of the particle as t approaches infinity?

15.4 Related rates

A claim was made in the first section of this chapter that 'the chain rule is the most important, and most widely used, rule of differentiation'. The chain rule has been repeatedly applied in all parts of this chapter thus far. Another important use of the chain rule is to find the rates of change of two or more variables that are changing with respect to time. Calculus provides us with the tools and techniques to solve problems where quantities (variables) are changing rather than static.

When a stone is thrown into a pond, a circular pattern of ripples is formed. In this situation we can observe an ever-widening circle moving across the water. As the circular ripple moves across the water, the radius r of the circle, its circumference C, and its area A all increase as a function of time t. Not only are these quantities (variables) functions of time, but their values at any particular time t are related to one another by familiar formulae such as $C = 2\pi r$ and $A = \pi r^2$. Thus their rates of change are also related to one another.

Example 20

A stone is thrown into a pond causing ripples in the form of concentric circles to move away from the point of impact at a rate of 20 cm per second. Find the following when a circular ripple has a radius of 50 cm and again when its radius is 100 cm.

a) the rate of change of the circle's circumference

b) the rate of change of the circle's area

Solution

In calculus, a derivative represents a rate of change of one variable with respect to another variable. If the circles are moving outward at a rate of 20 cm/sec, then the rate of change of the radius is 20 cm/sec, and in the notation of calculus we write

$$\frac{dr}{dt} = 20.$$

a) Knowing that the relationship between the radius, r, and the circumference, C, is $C = 2\pi r$, and that the rate of change of the radius with respect to time is $\frac{dr}{dt} = 20$, we can use the chain rule to find the rate of change of the circumference with respect to time, i.e. $\frac{dC}{dt}$.

$$\frac{dC}{dt} = \frac{dC}{dr} \cdot \frac{dr}{dt}$$

We need to find $\frac{dC}{dr}$, the rate of change (derivative) of the circumference with respect to the radius. This rate can be derived from the relationship between the variables.

$$C = 2\pi r$$

$$\frac{d}{dr}(C) = \frac{d}{dr}(2\pi r)$$ Differentiate both sides with respect to r.

$$\frac{dC}{dr} = 2\pi$$ Implicit differentiation on the left side.

Since the circumference C is a linear function of the radius r ($C = 2\pi r$), the derivative $\frac{dC}{dr}$ is a constant.

We now substitute in for $\frac{dC}{dr}$ and $\frac{dr}{dt}$ to find the rate of change of the circumference with respect to time, $\frac{dC}{dt}$.

$$\frac{dC}{dt} = \frac{dC}{dr} \cdot \frac{dr}{dt} \Rightarrow \frac{dC}{dt} = 2\pi \cdot 20 = 40\pi \text{ cm/sec}$$

The rate of change of a circular ripple's circumference is constant (40π). Therefore, the rate of change of the circumference is 40π cm/sec when the radius is 50 cm and also when its 100 cm.

• **Hint:** There is a slightly different method to determine $\frac{dC}{dt}$. We can find the rate by differentiating implicitly with respect to time, t, both sides of the equation, $C = 2\pi r$, that gives the relationship between the two changing quantities (variables).

$C = 2\pi r$

Differentiate both sides with respect to t:

$\frac{d}{dt}(C) = \frac{d}{dt}(2\pi r)$

Implicit differentiation:

$\frac{dC}{dt} = 2\pi \frac{dr}{dt}$

Substitute $\frac{dr}{dt} = 20$:

$\frac{dC}{dt} = 2\pi \cdot 20 = 40\pi$ cm/sec

b) Similarly, to find the rate of change of the area with respect to time, $\frac{dA}{dt}$, we can use the chain rule to write

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}.$$

Find $\frac{dA}{dr}$ from the formula, $A = \pi r^2$, that relates the variables A and r.

$$\frac{d}{dr}(A) = \frac{d}{dr}(\pi r^2)$$ Differentiate both sides with respect to r.

$$\frac{dA}{dr} = \pi(2r) = 2\pi r$$ Implicit differentiation on the left side.

Since the area A is a non-linear function of the radius r ($A = \pi r^2$), the derivative $\frac{dA}{dr}$ is not a constant but has different values depending on the value of r.

We substitute in for $\frac{dA}{dr}$ and $\frac{dr}{dt}$ to find the rate of change of the area with respect to time, $\frac{dA}{dt}$.

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} \Rightarrow \frac{dA}{dt} = 2\pi r \cdot 20 = 40\pi r$$

Thus, the rate of change of the circle's area with respect to time, $\frac{dA}{dt}$, is a linear function in terms of the radius r.

When the radius is 50 cm, $\frac{dA}{dt} = 40\pi \cdot 50 = 2000\pi\,\text{cm}^2/\text{sec}$
$\approx 6280\,\text{cm}^2/\text{sec}$ [$\approx 0.628\,\text{m}^2/\text{sec}$].

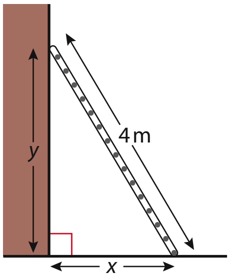
When the radius is 100 cm, $\frac{dA}{dt} = 40\pi \cdot 100 = 4000\pi\,\text{cm}^2/\text{sec}$
$\approx 12\,600\,\text{cm}^2/\text{sec}$ [$\approx 1.26\,\text{m}^2/\text{sec}$].

Note that when $r = 100$ cm the area is changing at twice the rate it was when $r = 50$ cm.

• **Hint:** It is important to include the appropriate units when giving a rate of change (derivative) answer. For example cm/sec, m^2/hour, litres/sec, etc.

Example 21

A 4-metre ladder stands upright against a vertical wall. If the foot of the ladder is pulled away from the wall at a constant rate of 0.75 m/sec, how fast is the top of the ladder coming down the wall at the instant it is (i) 3 metres above the ground, and (ii) 1 metre above the ground. Give answers approximate to three significant figures.



Solution

Let x and y represent the distances of the foot and top of the ladder, respectively, from the bottom of the wall. Then from Pythagoras' theorem, we have

$$x^2 + y^2 = 16.$$

Given that the ladder is being pulled away at a rate of 0.75 m/sec, then

$$\frac{dx}{dt} = 0.75 = \frac{3}{4}.$$

So we know the rate $\frac{dx}{dt}$, and we need to find $\frac{dy}{dt}$ when $y = 3$ and when $y = 1$.

Rather than starting with the chain rule and writing an equation relating the different rates, let's utilize the chain rule by differentiating implicitly with respect to time the equation relating the relevant variables x and y.

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(16)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

(i) We know $\frac{dx}{dt} = \frac{3}{4}$, so to find $\frac{dy}{dt}$ when $y = 3$ m, we find the corresponding value for x.

$$x^2 + y^2 = 16 \Rightarrow x = \sqrt{16 - y^2};\ \text{for } y = 3\text{: } x = \sqrt{16 - 3^2} = \sqrt{7}$$

Hence, when $y = 3$: $\frac{dy}{dt} = -\frac{\sqrt{7}}{3} \cdot \frac{3}{4} = -\frac{\sqrt{7}}{4} \approx -0.661$ m/sec.

(ii) For $y = 1$: $x = \sqrt{16 - 1^2} = \sqrt{15}$

Hence, when $y = 1$: $\frac{dy}{dt} = -\frac{\sqrt{15}}{1} \cdot \frac{3}{4} = -\frac{3\sqrt{15}}{4} \approx -2.90$ m/sec.

It makes sense that $\frac{dy}{dt}$ is negative because the distance y decreases as the ladders slides down.

Example 22

In the preceding example, how fast is the angle between the ladder and the ground changing when $y = 2$ m?

Solution

We know $\frac{dx}{dt} = \frac{3}{4}$ and we seek to find $\frac{d\theta}{dt}$. We need a relationship, true at any instant, between the variables θ and x. Several trigonometric ratios could be used, but perhaps the most straightforward is

$$x = 4\cos\theta.$$

Now we differentiate implicitly with respect to t and solve for $\frac{d\theta}{dt}$.

$$\frac{d}{dt}(x) = \frac{d}{dt}(4\cos\theta)$$

$$\frac{dx}{dt} = -4\sin\theta\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{4\sin\theta}\frac{dx}{dt}$$

When $y = 2$ we find that $\sin\theta = \frac{1}{2}$. Substituting appropriately for $\sin\theta$ and $\frac{d\theta}{dt}$, we have

$$\frac{d\theta}{dt} = -\frac{1}{4\left(\frac{1}{2}\right)} \cdot \frac{3}{4} = -\frac{3}{8}.$$

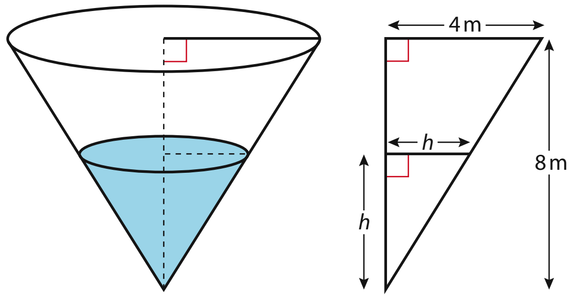
Therefore, the angle is decreasing at a rate of $\frac{3}{8}$ radians/sec (or approximately 21.5°/sec).

The solution strategy used in the preceding two examples is summarized below.

Solving problems involving related rates

1. Identify any rate(s) of change you know and the rate of change to be found.
2. Draw a diagram with all of the important information clearly labelled.
3. Write an equation relating the variables whose rates of change are either known or are to be found.
4. Using the chain rule, differentiate the equation implicitly with respect to time. Solve for the rate to be found.
5. Substitute in all known values for any variables and any rates of change. Compute the required rate of change. Be sure to include appropriate units with the result.

Example 23



Consider a conical tank as shown in the diagram. Its radius at the top is 4 metres and its height is 8 metres. The tank is being filled with water at a rate of 2 m^3/min. How fast is the water level rising when it is 5 metres high?

Solution

We know the rate of change of the volume with respect to time, that is, $\frac{dV}{dt} = 2$ m^3/min and we seek to find the rate of change of the height of the water level with respect to time, call it $\frac{dh}{dt}$.

Not including t, there are three variables involved in this problem: V, r and h. The formula for the volume of a cone will give us an equation that relates all of these variables.

$$V = \tfrac{1}{3}\pi r^2 h$$

If we differentiate this equation now we will get the rate $\frac{dr}{dt}$ in our result. We need to either find $\frac{dr}{dt}$ (which is possible) or eliminate r from the equation by solving for it in terms of one of the other variables and substitute. By using similar triangles we can write a proportion involving r and h.

$$\frac{r}{h} = \frac{4}{8} \Rightarrow r = \frac{h}{2}$$

Hence, $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \Rightarrow V = \frac{\pi}{12}h^3$.

Differentiating implicitly with respect to t and solving for $\frac{dh}{dt}$:

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2\frac{dh}{dt} \Rightarrow \frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 5$ and $\frac{dV}{dt} = 2$ gives

$$\frac{dh}{dt} = \frac{4}{\pi(5)^2} \cdot 2 = \frac{8}{25\pi} \approx 0.102\text{ m/min [or 10.2 cm/min]}.$$

Therefore, the water level is rising at a rate of 0.102 m/min when the water level is at 5 m.

• **Hint:** Be careful not to substitute in known quantities too early in the process of solving a related rates problem. Substitute the known values of any variables and any rates of change *after* differentiation. For example, in Example 23 h remained a variable (it is a quantity that is changing over time) until the last stage of the solution when we substituted $h = 5$. If we substituted earlier into $V = \frac{\pi}{12}h^3$, we would have obtained $\frac{dV}{dt} = 0$, which is obviously wrong.

The following example involves two rates of change.

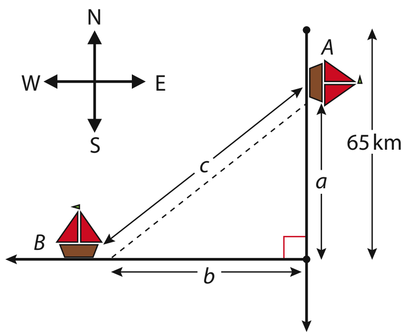
Example 24

At 12 noon ship A is 65 km due north of a second ship, B. Ship A sails south at a rate of 14 km/hr, and ship B sails west at a rate of 16 km/hr.

a) How fast are the two ships approaching each other $1\frac{1}{2}$ hours later at 1:30?

b) At what time do the two ships stop approaching and begin moving away from each other?

Solution



Let a and b be the distances that ships A and B, respectively, are from the intersection of the ships' paths (see diagram). Let c be the distance between the two ships. Since a is decreasing and b is increasing, we know that $\frac{da}{dt} = -14$ km/hr and $\frac{db}{dt} = 16$ km/hr.

a) The three variables are related by the equation

$$c^2 = a^2 + b^2.$$

Differentiating implicitly with respect to t gives

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt}.$$

The rate at which the ships are approaching is $\frac{dc}{dt}$. Solving for $\frac{dc}{dt}$:

$$\frac{dc}{dt} = \frac{a\frac{da}{dt} + b\frac{db}{dt}}{c}$$

Substituting $\frac{da}{dt} = -14$ and $\frac{db}{dt} = 16$:

$$\frac{dc}{dt} = \frac{-14a + 16b}{c}$$

The distances a and b are both functions of time; thus, they can be written in terms of t as

$$a = 65 - 14t \text{ and } b = 16t.$$

Evaluating these expressions when $t = 1\frac{1}{2}$, gives $a = 44$, $b = 24$ and $c = \sqrt{44^2 + 24^2} \approx 50.12$. Substituting these values into the expression for $\frac{dc}{dt}$ gives

$$\frac{dc}{dt} \approx \frac{-14(44) + 16(24)}{50.12} \approx -4.629.$$

Therefore, at 1:30 the distance between the two ships is decreasing at a rate of approximately -4.63 km/hr.

b) The time at which the two ships will stop approaching each other and begin to move away is when the value of $\frac{dc}{dt}$ changes from negative to positive. So we need to find when $\frac{dc}{dt} = 0$.

$$\frac{dc}{dt} = \frac{-14a + 16b}{c} = 0 \Rightarrow -14a + 16b = 0$$

Substituting in $a = 65 - 14t$ and $b = 16t$ gives:

$$-14(65 - 14t) + (16t) = 0 \Rightarrow 452t - 910 = 0 \Rightarrow t = \frac{910}{452} \approx 2.013$$

Therefore, just moments after 2:00 the two ships will stop approaching and start moving away from each other.

Exercise 15.4

1 A water tank is in the shape of an inverted cone. Water is being drained from the tank at a constant rate of 2 m^3/min. (Since volume is decreasing, $\frac{dV}{dt}$ is negative.) The height of the tank is 8 m, and the diameter of the top of the tank is 6 m. When the height of the water is 5 m, find, in units of cm/min, the following:

a) the rate of change of the water level

b) the rate of change of the radius of the surface of the water.

2 A spherical balloon is being inflated at a constant rate of 240 cm^3/sec. [$V = \frac{4}{3}\pi r^3$]

a) At what rate is the radius increasing when the radius is equal to 8 cm?

b) At what rate is the radius increasing 5 seconds after the start of inflation?

3 Oil is dripping from a car engine on to a garage floor, making a growing circular stain. The radius, r, of the stain is increasing at a constant rate of 1 cm/hr. When the radius is 4 cm, find:

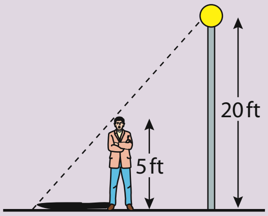
a) the rate of change of the circumference of the stain

b) the rate of change of the area of the stain.

4 A hot air balloon is rising straight up from a level field at a constant rate of 50 m/min. An observer is standing 150 m from the point on the ground where the balloon was launched. Let θ be the angle between the ground and the observer's line of sight to the balloon from the point at which the observer is standing (angle of elevation of the balloon). What is the rate of change of θ (in radians/min) when the height of the balloon is 250 m?

5 Jenny is flying a kite at a constant height above level ground of 72 m. The wind carries the kite away horizontally at a rate of 6 m/sec. How fast must Jenny let out the string at the moment when the kite is 120 m away from her?

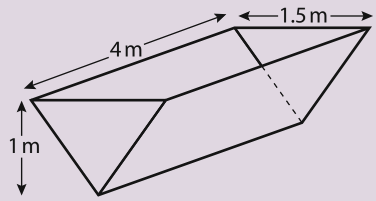
6 A 5-foot boy is walking toward a 20-foot lamp post at a constant rate of 6 ft/sec. The light from the lamp post causes the boy to cast a shadow. How fast is the tip of his shadow moving?



7 Two cars start from a point A at the same time. One travels west at 60 km/hr and the other travels north at 35 km/hr. How fast is the distance between them increasing 3 hours later?

8 A point moves along the curve $y = \sqrt{x^2 + 1}$ in such a way that $\frac{dx}{dt} = 4$. Find $\frac{dy}{dt}$ when $x = 3$.

9 A horizontal trough is 4 m long, 1.5 m wide and 1 m deep. Its cross-section is an isosceles triangle. Water is flowing into the trough at a constant rate of 0.03 m^3/sec. Find the rate at which the water level is rising 25 seconds after the water started flowing into the trough.



10 If the radius of a sphere is increasing at the constant rate of 3 mm/sec, how fast is the volume changing when the surface area is 10 mm^2? [Surface area $= 4\pi r^2$]

11 Two roads, A and B, intersect each other at an angle of 60°. Two cars, one on road A travelling at 40 km/hr and the other on road B travelling at 50 km/hr, are approaching the intersection. If, at a certain moment, the two cars are both 2 km from the intersection, how fast is the distance between them changing?

12 If the diagonal of a cube is increasing at a rate of 8 cm/sec, how fast is a side of the cube increasing?

13 A point P is moving along the circle with equation $x^2 + y^2 = 100$ at a constant rate of 3 units/sec. How fast is the projection of P on the x-axis moving when P is 5 units above the x-axis?

14 A jet is flying at a constant speed at an altitude of 10 000 m on a path that will take it directly over an observer on the ground. At a given instant the observer determines that the angle of elevation of the jet is $\frac{\pi}{3}$ radians and is increasing at a constant rate of $\frac{1}{60}$ radians/sec. Find the speed of the jet.

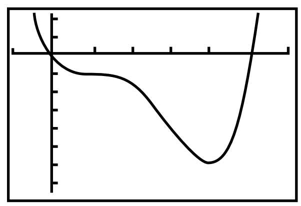
15 A television cameraman is filming an automobile race from a platform that is 40 metres from the racing track, following a car that is moving at 288 km/hr. How fast, in degrees per second, will the camera be turning when a) the car is directly in front of the camera and b) a half second later? Answer to the nearest whole degree.

16 A plane is flying due east at 640 km/hr and climbing vertically at a rate of 180 m/min. An airport tower is tracking it. Determine how fast the distance between the plane and the tower is changing when the plane is 5 km above the ground over a point exactly 6 km due west of the tower. Express the answer in km/hr.

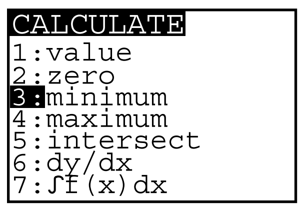
15.5 Optimization

Many problems in science and mathematics involve finding the maximum or minimum value (**optimum** value) of a function over a specified or implied domain. The development of the calculus in the seventeenth century was motivated to a large extent by maxima and minima (**optimization**) problems. One such problem lead Pierre de Fermat (1601–1665) to develop his Principle of Least Time: a ray of light will follow the path that takes the least (or minimum) time. The solution to Fermat's principle lead to Snell's law, or law of refraction (see the investigation at the end of this section). The solution is found by applying techniques of differential calculus – which can also be used to solve other optimization problems involving ideas such as least cost, maximum profit, minimum surface area and greatest volume.

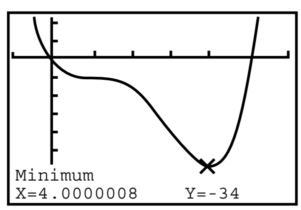
Previously, we learned the theory of how to use the derivative of a function to locate points where the function has a maximum or minimum (i.e. extreme) value. It is important to remember that if the derivative of a function is zero at a certain point it does not *necessarily* follow that the function has an extreme value (relative or absolute) at that point – it only ensures that the function has a horizontal tangent (stationary point) at that point. An extreme value *may* occur where the derivative is zero or at the endpoints of the function's domain.



The graph of $f(x) = x^4 - 8x^3 + 18x^2 - 16x - 2$ is shown left. The derivative of $f(x)$ is $f'(x) = 4x^3 - 24x^2 + 36x - 16 = 4(x - 4)(x - 1)^2$. The function has horizontal tangents at both $x = 1$ and $x = 4$, since the derivative is zero at these points. However, an extreme value (absolute minimum) occurs only at $x = 4$. It is important to confirm – graphically (see GDC images) or algebraically – the precise nature of a point on a function where the derivative is zero. Some different algebraic methods for confirming that a value is a maximum or minimum will be illustrated in the examples that follow.



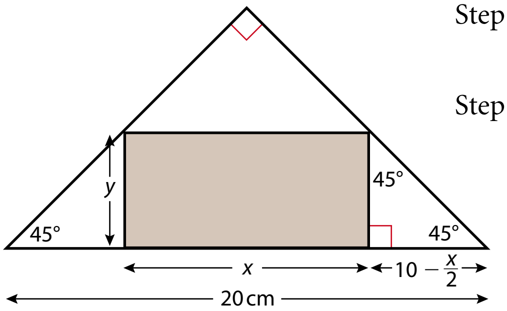
It is also useful to remember that one can often find extreme values (extrema) without calculus (e.g. using a 'minimum' command on a graphics calculator, as shown). Calculator or computer technology can be very helpful in modelling, solving or confirming solutions to optimization problems. However, it is important to learn how to apply algebraic methods of differentiation to optimization problems because it may be the only efficient way to obtain an accurate solution.



Let's start with a relatively straightforward example. We can use the steps in the solution to develop a general strategy that can be applied to more sophisticated problems.

Example 25 – Finding a maximum area (Developing a general strategy)

Find the maximum area of a rectangle inscribed in an isosceles right triangle whose hypotenuse is 20 cm long.



Solution

Step 1: Draw an accurate diagram. Let the base of the rectangle be x cm and the height y cm. Then the area of the rectangle is $A = xy\,\text{cm}^2$.

Step 2: Express area as a function in terms of only one variable.

It can be deduced from the diagram that $y = 10 - \frac{x}{2}$.

Therefore, $A(x) = x\left(10 - \frac{x}{2}\right) = 10x - \frac{x^2}{2}$.

x must be positive and from the diagram it is clear that x must be less than 20 (domain of A: $0 < x < 20$).

Step 3: Find the derivative of the area function and find for what value(s) of x it is zero.

$A'(x) = 10 - x$ $\qquad$ $A'(x) = 0$ when $x = 10$

Step 4: Analyze $A(x)$ at $x = 10$ and also at the endpoints of the domain, $x = 0$ and $x = 20$.

The second derivative test (Section 13.3) provides information about the concavity of a function. The second derivative is $A''(x) = -1$ and since $A''(x)$ is always negative then $A(x)$ is always concave down, indicating $A(x)$ has a maximum at $x = 10$.

$A(0) = 0$ and $A(20) = 0$, indicating $A(x)$ has an absolute maximum at $x = 10$.

Therefore, the rectangle has a maximum area equal to

$A(10) = 10\left(10 - \frac{10}{2}\right) = 50\,\text{cm}^2$.

General strategy for solving optimization problems

Step 1: Draw a diagram that accurately illustrates the problem. Label all known parts of the diagram. Using variables, label the important unknown quantity (or quantities) (for example, x for base and y for height in Example 25).

Step 2: For the quantity that is to be optimized (area in Example 25), express this quantity as a function in terms of a single variable. From the diagram and/or information provided, determine the domain of this function.

Step 3: Find the derivative of the function from Step 2, and determine where the derivative is zero. This value (or values) of the derivative, along with any domain endpoints, are the **critical values** ($x = 0$, $x = 10$ and $x = 20$ in Example 25) to be tested.

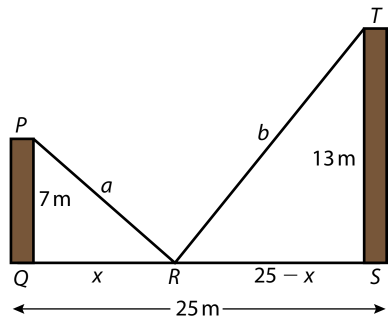
Step 4: Using algebraic (e.g. second derivative test) or graphical (e.g. GDC) methods, analyze the nature (maximum, minimum, neither) of the points at the critical values for the optimized function. Be sure to answer the precise question that was asked in the problem.

Example 26 – Finding a minimum length – two posts problem

Two vertical posts, with heights of 7 m and 13 m, are secured by a rope going from the top of one post to a point on the ground between the posts and then to the top of the other post. The distance between the two posts is 25 m. Where should the point at which the rope touches the ground be located so that the least amount of rope is used?

Solution

Step 1: An accurate diagram is drawn. The posts are drawn as line segments PQ and TS and the point where the rope touches the ground is labelled R. The optimum location of point R can be given as a distance from the base of the shorter post, QR, or from the taller post, SR. It is decided to give the answer as the distance from the shorter post – and this is labelled x. There are two other important unknown quantities: the lengths of the two portions of the rope, PR and TR. These are labelled a and b, respectively.



Step 2: The quantity to be minimized is the length L of the rope, which is the sum of a and b. From Pythagoras' theorem, $a = \sqrt{x^2 + 49}$ and $b = \sqrt{(25 - x)^2 + 169}$. Therefore, the function for length (L) can be expressed in terms of the single variable x as

$$L(x) = \sqrt{x^2 + 49} + \sqrt{(25 - x)^2 + 169}$$

$$= \sqrt{x^2 + 49} + \sqrt{x^2 - 50x + 625 + 169}$$

$$L(x) = \sqrt{x^2 + 49} + \sqrt{x^2 - 50x + 794}$$

From the given information and diagram, the domain of $L(x)$ is $0 \leqslant x \leqslant 25$.

Step 3: To facilitate differentiation, express $L(x)$ using fractional exponents:

$$L(x) = (x^2 + 49)^{\frac{1}{2}} + (x^2 - 50x + 794)^{\frac{1}{2}}$$

Then apply the chain rule for differentiation:

$$\frac{dL}{dx} = \frac{1}{2}(x^2 + 49)^{-\frac{1}{2}}(2x) + \frac{1}{2}(x^2 - 50x + 794)^{-\frac{1}{2}}(2x - 50) \Rightarrow$$

$$\frac{dL}{dx} = \frac{x}{\sqrt{x^2 + 49}} + \frac{x - 25}{\sqrt{x^2 - 50x + 794}}$$

By setting $\frac{dL}{dx} = 0$, we obtain

$$x\sqrt{x^2 - 50x + 794} = -(x - 25)\sqrt{x^2 + 49}$$

$$x^2(x^2 - 50x + 794) = (25 - x)^2(x^2 + 49)$$

$$x^4 - 50x^3 + 794x^2 = x^4 - 50x^3 + 674x^2 - 2450x + 30\,625$$

$$120x^2 + 2450x - 30\,625 = 0$$

$$5(4x - 35)(6x + 175) = 0$$

$$x = \frac{35}{4} \quad \text{or} \quad x = -\frac{175}{6}$$

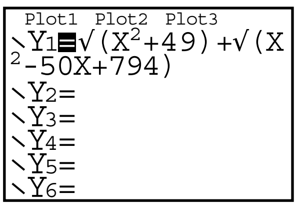
Step 4: Since $x = -\frac{175}{6}$ is not in the domain for $L(x)$, then the critical values are $x = 0$, $x = \frac{35}{4}$ and $x = 25$. Simply evaluate $L(x)$ for these critical values.

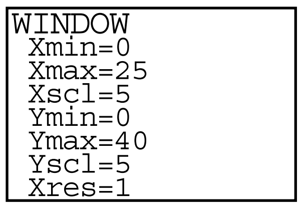
$L(0) = 7 + \sqrt{794} \approx 35.18$, $\quad L(25) = \sqrt{674} + 13 \approx 38.96$,

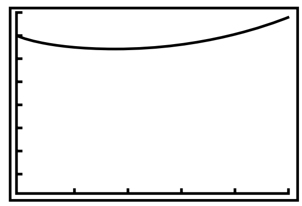
$L\left(\frac{35}{4}\right) = 5\sqrt{41} \approx 32.02$

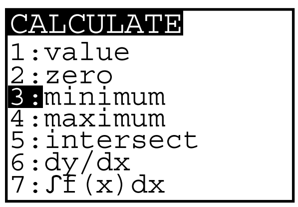
Therefore, the rope should touch the ground at a distance of $\frac{35}{4} = 8.75$ m from the base of the shorter post, to give a minimum rope length of approximately 32.02 m.

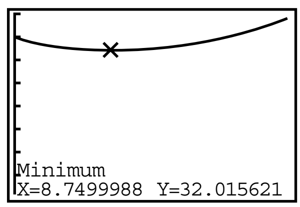
The minimum value could also be confirmed from the graph of $L(x)$, but it would be difficult to confirm using the second derivative test because of the algebra required. From this example, we can see that applied optimization problems can involve a high level of algebra. If you have access to suitable graphing technology, you could perform Steps 3 and 4 graphically rather than algebraically.











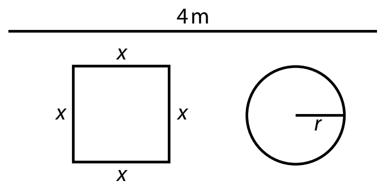
It is interesting to observe that the result for x produced by the calculator does not appear to be exact. Why is that? Algebraic techniques using differentiation give us the certainty of an exact solution while also allowing us to deal with the abstract nature of optimization problems involving parameters rather than fixed measurements (e.g. the heights of the posts).

In both Example 25 and 26, the extreme value occurred at a point where the derivative was zero. Although this often happens, an extreme value may occur at the endpoint of the domain.

Example 27 – An endpoint maximum

A supply of four metres of wire is to be used to form a square and a circle. How much of the wire should be used to make the square and how much should be used to make the circle in order to enclose the greatest amount of area? Guess the answer before looking at the following solution.

Solution



Step 1: Let x = length of each edge of the square and r = radius of the circle.

Step 2: The total area is given by $A = x^2 + \pi r^2$. The task is to write the area A as a function of a single variable. Therefore, it is necessary to express r in terms of x, or vice versa, and perform a substitution.

The perimeter of the square is $4x$ and the circumference of the circle is $2\pi r$. The total amount of wire is 4 m which gives

$$4 = 4x + 2\pi r \quad \Rightarrow \quad 2\pi r = 4 - 4x \quad \Rightarrow \quad r = \frac{2(1-x)}{\pi}$$

Substituting gives $A(x) = x^2 + \pi\left[\frac{2(1-x)}{\pi}\right]^2 = x^2 + \frac{4(1-x)^2}{\pi}$

$$= \frac{1}{\pi}[(\pi + 4)x^2 - 8x + 4]$$

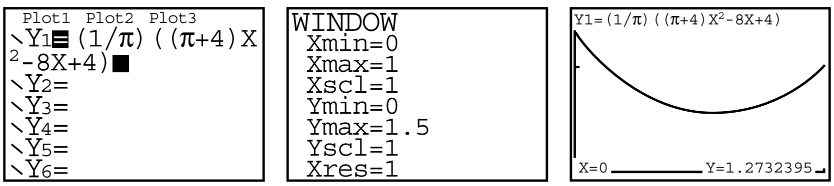
Because the square's perimeter is $4x$, then the domain for $A(x)$ is $0 \leqslant x \leqslant 1$.

Step 3: Differentiate the function $A(x)$, set equal to zero, and solve.

$$\frac{d}{dx}\left(\frac{1}{\pi}[(\pi + 4)x^2 - 8x + 4]\right) = \frac{1}{\pi}[2(\pi + 4)x - 8] = 0$$

$$2(\pi + 4)x - 8 = 0 \Rightarrow (\pi + 4)x = 4 \Rightarrow x = \frac{4}{\pi + 4} \approx 0.5601$$

The critical values are $x = 0$, $x \approx 0.5601$ and $x = 1$.

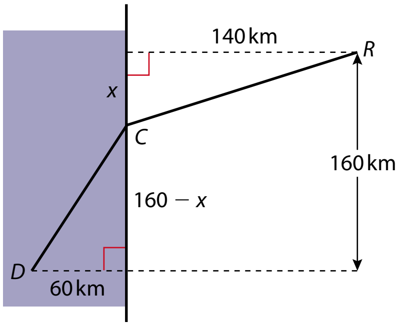


Step 4: Evaluating $A(x)$: $A(0) \approx 1.273$, $A(0.5601) \approx 0.5601$ and $A(1) = 1$. Therefore, the maximum area occurs when $x = 0$ which means all the wire is used for the circle.

What would the answer be if Example 27 asked for the dimensions of the square and circle to enclose the *least* total area?

Example 28 – Minimizing time

A pipeline needs to be constructed to link an offshore drilling rig to an onshore refinery depot. The oil rig is located at a distance (perpendicular to the coast) of 140 km from the coast. The depot is located inland at a distance (perpendicular) of 60 km from the coast. For modelling purposes, the coastline is assumed to follow a straight line. The point on the coastline nearest to the oil rig is 160 km from the point on the coastline nearest to the depot. The rate at which crude oil is pumped through the pipeline varies according to several variables, including pipe dimensions, materials, temperature, etc. On average, oil flows through the offshore section of the pipeline at a rate of 9 km per hour and 5 km per hour through the onshore section. Assume that both sections of pipeline can travel straight from one point to another. At what point should the pipeline intersect with the coastline in order for the oil to take a minimum amount of time to flow from the rig to the depot?



Solution

Step 1: The optimum location of the point, C, where the pipeline comes ashore will be designated by the distance, x, it is from the point on the coast that is a minimum distance (perpendicular) from the rig, R (140 km). The distance from R to C is $\sqrt{x^2 + 140^2}$ and the distance from D (depot) to C is $\sqrt{(160 - x)^2 + 60^2}$.

Step 2: The quantity to be minimized is time, so it is necessary to express the total time it takes the oil to flow from R to D in terms of a single variable.

$\text{time} = \dfrac{\text{distance}}{\text{rate}} \Rightarrow \text{time (offshore)} = \dfrac{\sqrt{x^2 + 19\,600} \text{ km}}{9 \text{ km/hr}}$;

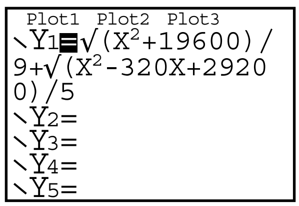
$\text{time (onshore)} = \dfrac{\sqrt{x^2 - 320x + 29\,200} \text{ km}}{5 \text{ km/hr}}$

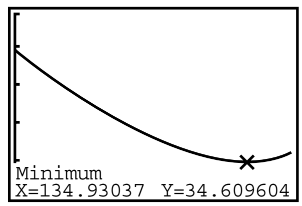
The function for time T in terms of x is:

$$T(x) = \frac{\sqrt{x^2 + 19\,600}}{9} + \frac{\sqrt{x^2 - 320x + 29\,200}}{5}$$

and the domain for $T(x)$ is $0 \leqslant x \leqslant 160$.

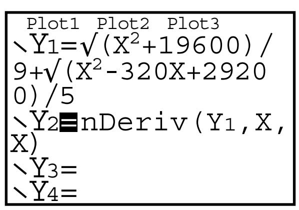
Steps 3/4: The algebra for finding the derivative of $T(x)$ is similar to that of Step 3 in Example 26. Let's use graphing technology to find the value of x that produces a minimum for $T(x)$.

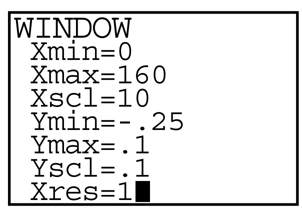


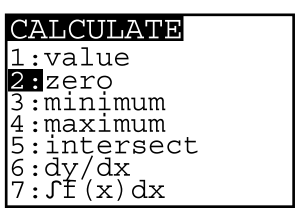


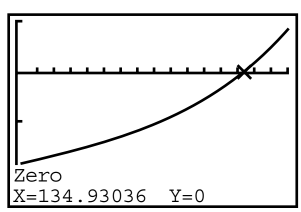
Therefore, the optimum point for the pipeline to intersect with the coast is approximately 134.9 km from the point on the coast nearest to the drilling rig.

The result could also be obtained by having a calculator or computer graph the derivative of $T(x)$ and compute any zeros for $T'(x)$ in the domain.





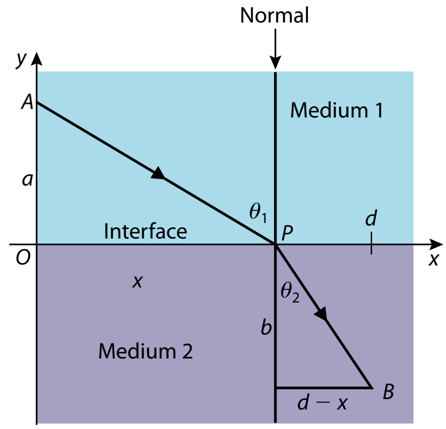




See the **Investigation** and how solving a problem similar to Example 28 derives Snell's law (or law of refraction).

Investigation – Snell's law

The speed of light depends on the medium through which light travels and is generally slower in denser media. The speed of light in a vacuum is an important physical constant and is exactly 299 792 458 m/s. A metre is defined to be the distance that light travels in a vacuum in $\frac{1}{299\,792\,458}$ of a second. Typically, the speed of light in a vacuum (denoted by the letter c) is given the approximate value of 3×10^8 m/s, but in the Earth's atmosphere light travels more slowly than that and even more slowly through glass and water.

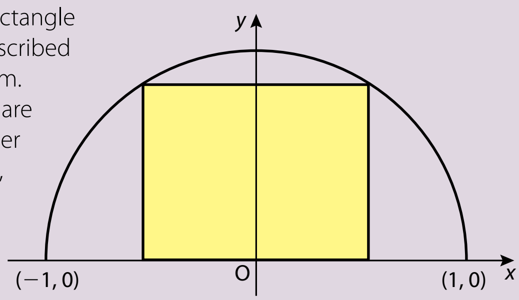


Fermat's principle in optics states that light travels from one point to another along a path for which time is a minimum. Investigate the path that a ray of light will follow in going from a point A in a transparent medium, where the speed of light is c_1, to a point B in a different transparent medium, where its speed is c_2, as illustrated in the diagram left. Using algebra and differentiation, prove that for time to be a minimum the following relationship must hold: $\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$. This equation is known as Snell's law or the law of refraction. Why is a graphics calculator not helpful?

Assume that the two points, A and B, lie in the xy-plane and the x-axis (interface) separates the two media. A light ray is refracted (deflected) when it passes from one medium to another. θ_1 is the **angle of incidence** and θ_2 is the **angle of refraction** (both angles measured between ray and normal to the interface).

Exercise 15.5

1 Find the dimensions of the rectangle with maximum area that is inscribed in a semicircle with radius 1 cm. Two vertices of the rectangle are on the semicircle and the other two vertices are on the x-axis, as shown in the diagram.



2 A rectangular piece of aluminium is to be rolled to make a cylinder with open ends (a tube). Regardless of the dimensions of the rectangle, the perimeter of the rectangle must be 40 cm. Find the dimensions (length and width) of the rectangle that gives a maximum volume for the cylinder.

3 Find the minimum distance from the graph of the function $y = \sqrt{x}$ and the point $(\frac{3}{2}, 0)$.

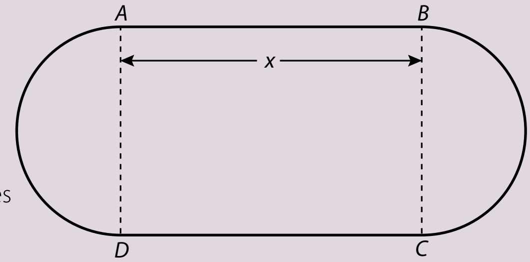
4 A rectangular box has height h cm, width x cm and length $2x$ cm. It is designed to have a volume equal to 1 litre (1000 cm^3).

a) Show that $h = \frac{500}{x^2}$ cm.

b) Find an expression for the total surface area, S cm^2, of the box in terms of x.

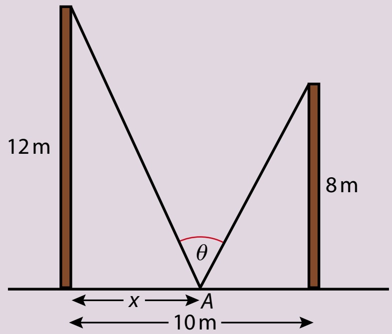
c) Find the dimensions of the box that produces a minimum surface area.

5 The figure right consists of a rectangle $ABCD$ and two semicircles on either end. The rectangle has an area of 100 cm^2. If x represents the length of the rectangle AB, find the value of x that makes the perimeter of the entire figure a minimum.

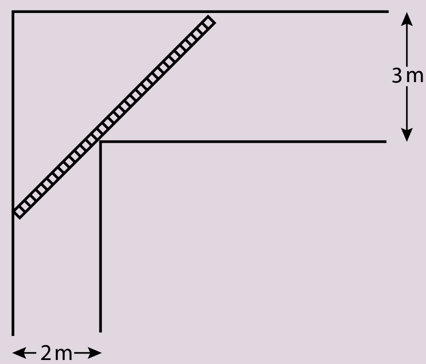


● **Hint:** Write an equation for θ in terms of x and find the value of x which makes θ a maximum by using your GDC.

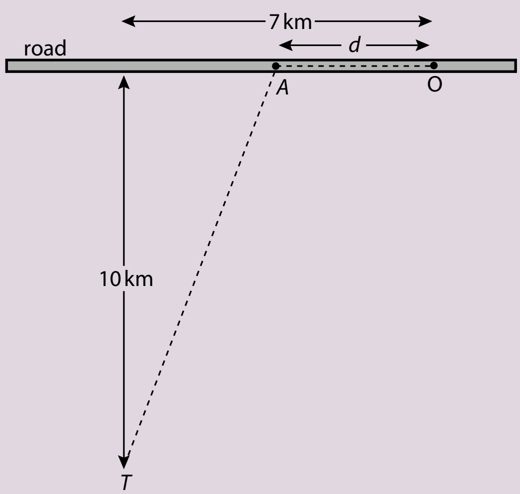
6 Two vertical posts, with heights 12 metres and 8 metres, are 10 metres apart on horizontal ground. A rope that stretches is attached to the top of both posts and is stretched down so that it touches the ground at point A between the two posts. The distance from the base of the taller post to point A is represented by x and the angle between the two sections of rope is θ. What value of x makes θ a maximum?



7 A ladder is to be carried horizontally down an L-shaped hallway. The first section of the hallway is 2 metres wide and then there is a right-angled turn into a 3-metre wide section. What is the longest ladder that can be carried around the corner?



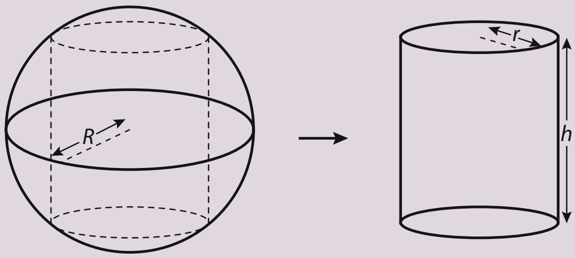
8 Charlie is walking from the wildlife observation tower (point T) to the Big Desert Park office (point O). The tower is 7 km due west and 10 km due south from the office. There is a road that goes to the office that Charlie can get to if she walks 10 km due north from the tower. Charlie can walk at a rate of 2 kilometres per hour (kph) through the sandy terrain of the park, but she can walk a faster rate of 5 kph on the road. To what point, A, on the road should Charlie walk to in order to take the least time to walk from the tower to the office? Find the value of d such that point A is d km from the office.



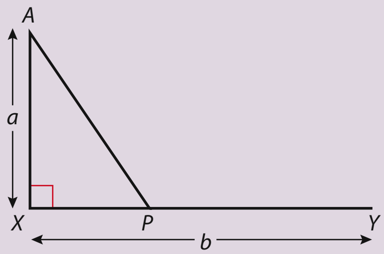
9 Two vertices of a rectangle are on the x-axis, and the other two vertices are on the curve $y = \dfrac{8}{x^2 + 4}$. (See Exercise 15.1, question 12.) Find the maximum area of the rectangle.

10 A ship sailing due south at 16 knots is 10 nautical miles north of a second ship going due west at 12 knots. Find the minimum distance between the two ships.

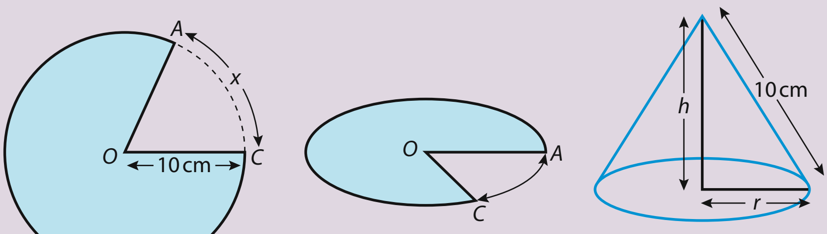
11 Find the height, h, and the base radius, r, of the largest right circular cylinder that can be made by cutting it away from a sphere with a radius of R.



12 Nadia is standing at point A that is a km away in the countryside from a straight road XY (see diagram). She wishes to reach the point Y where the distance from X to Y is b km. Her speed on the road is r km/hr and her speed travelling across the countryside is c km/hr, such that $r > c$. If she wishes to reach Y as quickly as possible, find the position of point P where she joins the road.

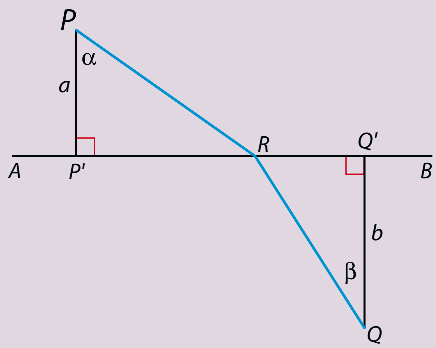


13 A cone of height h and radius r is constructed from a circle with radius 10 cm by removing a sector AOC of arc length x cm and then connecting the edges OA and OC. What arc length x will produce the cone of maximum volume, and what is the volume?



14 Point P is a units above the line AB, and point Q is b units below line AB (see diagram). The velocity of light is u units/second above AB and v units/second below AB, and $u > v$. The angles α and β are the angles that a ray of light makes with a perpendicular (normal) to line AB above and below AB, respectively. Show that the following relationship must hold true.

$$\frac{\sin\alpha}{\sin\beta} = \frac{u}{v}$$



13 a) $a=\frac{\pi}{2}, b=\pi, c=\frac{3\pi}{2}$

b) $\frac{d^{(n)}}{dx^{(n)}}(\sin x)=\sin\left(x+n\cdot\frac{\pi}{2}\right),\ n\in\mathbb{Z}^+$

14 a) $\frac{d}{dx}(xe^x)=xe^x+e^x$; $\frac{d^2}{dx^2}(xe^x)=xe^x+2e^x$;

$\frac{d^3}{dx^3}(xe^x)=xe^x+3e^x$

b) $\frac{d^{(n)}}{dx^{(n)}}(xe^x)=xe^x+ne^x$

Exercise 15.3

1 $\frac{dy}{dx}=-\frac{x}{y}$

2 $\frac{dy}{dx}=\frac{-2xy-y^2}{x^2+2xy}$

3 $\frac{dy}{dx}=\cos^2 y\ \left[\text{or } \frac{dy}{dx}=\frac{1}{1+x^2}\right]$

4 $\frac{dy}{dx}=\frac{-2x+3y^2-y^3}{-6xy+3xy^2-2y}$

5 $\frac{dy}{dx}=\frac{x^2y+y^3}{x^3+xy^2}$

6 $\frac{dy}{dx}=\frac{-2xy-2y^2-xy}{2x^2+2xy+xy}$

7 $\frac{dy}{dx}=\frac{y-1}{\cos y-x}$

8 $\frac{dy}{dx}=\frac{4x^3-2xy^3}{3x^2y^2+4y^3}$

9 $\frac{dy}{dx}=\frac{-y}{x+e^y}$

10 $\frac{dy}{dx}=\frac{x+2}{y+3}$

11 $\frac{dy}{dx}=-\sin^2(x+y)\ \left[\text{or } \frac{dy}{dx}=-\frac{x^2}{x^2+1}\right]$

12 $\frac{dy}{dx}=\frac{18x^2\sqrt{xy}-y}{x+2\sqrt{xy}}$

13 $y=-\frac{7}{5}x+\frac{4}{5}$; $y=\frac{5}{7}x-\frac{24}{7}$

14 $y=-2x+4$; $y=\frac{1}{2}x+\frac{3}{2}$

15 $y=-\frac{\pi}{2}x+\pi$; $y=\frac{2}{\pi}x+\frac{\pi^2-4}{2\pi}$

16 $y=-\frac{352}{23}x-\frac{32}{23}$; $y=\frac{23}{352}x-\frac{5655}{176}$

17 $x^2+y^2=r^2 \Rightarrow \frac{dy}{dx}=-\frac{x}{y}$; at point (x_1, y_1), $m=-\frac{x_1}{y_1}$; centre of circle is $(0,0)$; slope of line through (x_1, y_1) and $(0,0)$ is $\frac{y_1}{x_1}$; because $-\frac{x_1}{y_1}\times\frac{y_1}{x_1}=-1$, the tangent to the circle at (x_1, y_1) and the line through (x_1, y_1) and $(0,0)$ are perpendicular

18 a) $(\sqrt{7},0)$, $(-\sqrt{7},0)$; $\frac{dy}{dx}=\frac{-2x-y}{x+2y}$, at both points $\frac{dy}{dx}=-2$

b) $\left(\sqrt{\frac{7}{3}},-2\sqrt{\frac{7}{3}}\right)$ and $\left(-\sqrt{\frac{7}{3}},2\sqrt{\frac{7}{3}}\right)$

c) $\left(2\sqrt{\frac{7}{3}},-\sqrt{\frac{7}{3}}\right)$ and $\left(-2\sqrt{\frac{7}{3}},\sqrt{\frac{7}{3}}\right)$

19 $(0,0)$

20 $\frac{dy}{dx}=-\frac{4x}{9y}$, $\frac{d^2y}{dx^2}=\frac{-36y^2-16x^2}{81y^3}$

21 $\frac{dy}{dx}=\frac{2-y}{(x+3)^2}$, $\frac{d^2y}{dx^2}=\frac{2y-4}{(x+3)^2}$

22 a) $\frac{dy}{dx}=\frac{-1}{3x^{\frac{4}{3}}}$, $\frac{d^2y}{dx^2}=\frac{4}{9x^{\frac{7}{3}}}$

b) $\frac{dy}{dx}=-\frac{y}{3x}$, $\frac{d^2y}{dx^2}=\frac{4y}{9x^2}$

23 $y=x+\frac{1}{2}$

24 $\frac{dy}{dx}=\frac{3x^2}{x^3+1}$

25 $\frac{dy}{dx}=\cot x$

26 $\frac{dy}{dx}=\frac{x}{(x^2-1)\ln 5}$

27 $\frac{dy}{dx}=\frac{-1}{x^2-1}$

28 $\frac{dy}{dx}=\frac{1}{2x\ln 10\sqrt{\log x}}$

29 $\frac{dy}{dx}=\frac{2a}{x^2-a}$

30 $\frac{dy}{dx}=-\sin x$

31 $\frac{dy}{dx}=\frac{-1}{x\ln 3(\log_3 x)^2}$

32 $\frac{dy}{dx}=\ln x$

33 0

34 $y=\left(\frac{1}{8\ln 2}\right)x-\frac{1}{\ln 2}+3$

35 Verify

36 $x=\frac{1}{e^{\frac{3}{2}}}$

37 a) $g'(x)=\frac{1-\ln x}{x^2}$, $g''(x)=\frac{-3+2\ln x}{x^3}$

b) $g'(x)=0$ only at $x=e$; $g''(e)=-\frac{1}{e^3}<0$, $\therefore$ abs. max. at $x=e$, max. value of g is $\frac{1}{e}$

38 $\frac{dy}{dx}=\frac{1}{x^2+2x+2}$

39 $\frac{dy}{dx}=\frac{1}{x^2+1}$

40 $\frac{dy}{dx}=\frac{6}{x\sqrt{x^4-9}}$

41 $\frac{dy}{dx}=\left(\tan^{-1}x+\frac{x}{x^2+1}\right)e^{x\tan^{-1}x}$

42 $f'(x)=0$; the graph of $f(x)$ is horizontal

43 Verify

44 $y=\left(\frac{\pi+4}{2}\right)x+\frac{\pi-4}{4}$

45 a) For $0\leqslant x<\pi$, $f'(x)=-1$, therefore $f(x)$ is linear

b) $y=-x+\frac{\pi}{2}$

46 $\sqrt{10}\approx 3.16$ m

47 a) $\frac{1}{4}\,\text{m s}^{-1}$, $\frac{1}{20}\,\text{m s}^{-1}$

b) $-\frac{1}{4}\,\text{m s}^{-2}$, $-\frac{13}{800}\,\text{m s}^{-2}$

c) The particle initially is moving very fast to the right and then gradually slows down while continuing to move to the right.

d) $\lim_{t\to\infty} s(t)=\frac{\pi}{2}$ m

Exercise 15.4

1 a) -18.1 cm/min b) -6.79 cm/min

2 a) 0.298 cm/sec b) 0.439 cm/sec

3 a) 2π cm/hr b) 8π cm/hr

4 $\frac{d\theta}{dt}=\frac{3}{34}\approx 0.0882$ radians/min

5 26.4 m/sec

6 2 ft/sec

7 69.6 km/hr

8 $\frac{dy}{dt}=\frac{12}{\sqrt{10}}\approx 3.79$

9 0.01 m/sec

10 30 mm^3/sec

11 45 km/hr

12 $\frac{8\sqrt{3}}{3}\approx 4.62$ cm/sec

13 1.5 units/sec

14 $222.\dot{2}$ m/sec = 800 km/hr

15 a) 115 degrees/sec b) 57 degrees/sec

16 -485 km/hr

Exercise 15.5

1 $\sqrt{2}$ by $\frac{\sqrt{2}}{2}$

2 $13\frac{1}{3}$ cm by $6\frac{2}{3}$ cm

3 $\frac{\sqrt{5}}{2}$

4 b) $S = 4x^2 + \frac{3000}{x}$ c) 7.21 cm × 14.4 cm × 9.61 cm

5 $x = 5\sqrt{2\pi} \approx 12.5$ cm

6 $x \approx 3.62$ m

7 Longest ladder ≈ 7.02 m

8 $d \approx 2.64$ km

9 $\frac{8}{5}$ units^2

10 6 nautical miles

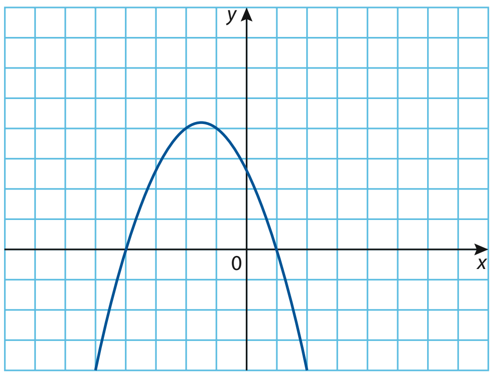
11 $h = R\sqrt{2}$, $r = \frac{R\sqrt{2}}{2}$

12 Distance of point P from point X is $\frac{ac}{\sqrt{r^2 - c^2}}$

13 $x \approx 51.3$ cm, maximum volume ≈ 403 cm^3

Practice questions

1



2 a) (i) $a = -4$ (ii) $b = 2$

(i) $f'(x) = -3x^2 - 4x + 8$

(ii) $\frac{-2+2\sqrt{7}}{3}$, $\frac{-2-2\sqrt{7}}{3}$

(iii) $f(1) = 5$

c) (i) $y = 8x$ (ii) $x = -2$

3 a) (i) $v(0) = 0$ (ii) $v(10) \approx 51.3$

b) (i) $a(t) = 0.99e^{-0.15t}$ (ii) $a(0) = 0.99$

c) (i) 66 (ii) 0

(iii) As object falls it approaches terminal velocity

4 a) $\left(-\frac{2}{3}, -\frac{149}{27}\right)$ is a minimum, $(-4, 13)$ is a maximum

b) $\left(-\frac{7}{3}, \frac{101}{27}\right)$ is an inflexion point

5 a) (i) $g'(x) = -\frac{3}{e^{3x}}$

(ii) $e^{3x} > 0$ for all x, hence $-\frac{3}{e^{3x}} < 0$ for all x; therefore, $f(x)$ is decreasing for all x

b) (i) $e + 2$

(ii) $g'(-\frac{1}{3}) = -3e$

c) $y = -3ex + 2$

6 b) $f'(3) = 0$ and $f''(3) > 0 \Rightarrow$ stationary point at $x = 3$ and graph of f is concave up at $x = 3$, so $f(3)$ is a minimum

c) $(4, 0)$

7 a) $-\frac{4}{(2x+3)^3}$

b) $5\cos(5x)e^{\sin(5x)}$

8 $A = 1$, $B = 2$, $C = 1$

9 $\frac{dy}{dx} = -1$, $\frac{d^2y}{dx^2} = -4$

10 a) $\frac{dy}{dx} = \frac{-xe^x + e^x - 1}{(e^x - 1)^2}$

b) $\frac{dy}{dx} = 2e^x\cos(2x) + e^x\sin(2x)$

c) $\frac{dy}{dx} = 2x\ln x + 2x\ln 3 + x - \frac{1}{x}$

11 $y = -\frac{1}{2}x - \frac{3}{2}$, $P(-3, 0)$, $Q\left(0, -\frac{3}{2}\right)$

12 a) $x = 3$; sign of $h''(x)$ changes from negative (concave down) to positive (concave up) at $x = 3$

b) $x = 1$; $h'(x)$ changes from positive (h increasing) to negative (h decreasing) at $x = 1$

13 $y = \frac{5}{7}x + \frac{11}{7}$

14 $h = 8$ cm, $r = 4$ cm

15 Maximum area is 32 square units; dimensions are 4 by 8

16 a) E b) A c) C

17 $y = -\frac{1}{5}x + \frac{32}{5}$

18 a) $y = 4x - 4$

b) $y = -\frac{1}{4}x + \frac{1}{4}$

19 a) Absolute minimum at $\left(\frac{1}{\sqrt{e}}, -\frac{1}{2e}\right)$

b) Inflexion point at $\left(\frac{1}{\sqrt{e^3}}, -\frac{3}{2e^3}\right)$

20 a) (i) $a = 16$ (ii) $a = 54$

b) $f'(x) = 2x - \frac{a}{x^2} = 0 \Rightarrow x = \sqrt[3]{\frac{a}{2}}$;

$f''(x) = 2 + \frac{2a}{x^3} \Rightarrow f''\left(\sqrt[3]{\frac{a}{2}}\right) = 4 > 0$; hence, f is concave up at any critical point, so it cannot be a maximum

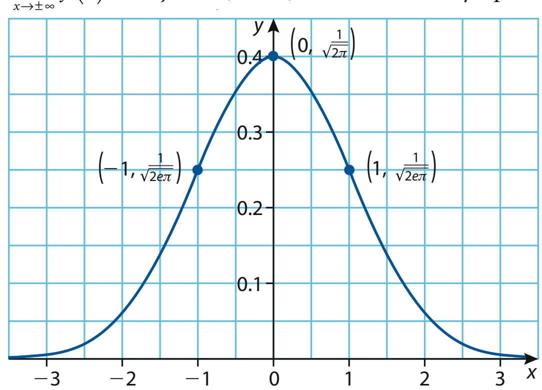
21 $y = -\frac{2}{3}x + 4$

22 $y = \left(\frac{\pi+2}{2}\right)x - \frac{\pi^2}{8}$; $y = \left(\frac{-2}{\pi+2}\right)x + \frac{\pi}{2\pi+4} + \frac{\pi}{4}$

23 a) Maximum at $\left(0, \frac{1}{\sqrt{2\pi}}\right)$, inflexion points at $\left(-1, \frac{1}{\sqrt{2e\pi}}\right)$ and $\left(1, \frac{1}{\sqrt{2e\pi}}\right)$

b) $\lim_{x \to \pm\infty} f(x) = 0$; $y = 0$ (x-axis) is a horizontal asymptote

c)



24 a) Min. at $x = 1$ because $f''(1) = \frac{1}{2} > 0$; max. at $x = 3$ because $f''(3) = -\frac{1}{6} < 0$

b) Inflexion points at $x = -\sqrt{3}$ and $x = \sqrt{3}$ because $f''(x)$ changes sign at both values