1. [5 *points*]

An **open** tank in the shape of a square-based cuboid is to be constructed. The volume of the tank must be 20 l. The cost of the base is 5 PLN per dm^2 and the cost of the side faces is 1 PLN per dm^2. Find the dimensions of the tank that minimize the cost.

Let the lenght of the side of the base be x and the height be h. We have:

$$x^2h = 20$$

Which gives $h = \frac{20}{x^2}$. We must minimize the cost:

$$C = 5x^2 + 4xh$$

The area of the base is x^2 times the cost, the area of the side faces is $4xh$ times the cost. We can express C in terms of x only:

$$C(x) = 5x^2 + \frac{80}{x}$$

with $x > 0$. Taking the derivative we get:

$$\frac{dC}{dx} = 10x - \frac{80}{x^2}$$

$$\frac{d^2C}{dx^2} = 10 + \frac{160}{x^3}$$

We have $\frac{dC}{dx} = 0$ for $x = 2$ and for this value of x the second derivative is positive, so this is a minimum. When $x = 2$, we get $h = 5$.

2. [5 *points*]

A cylinder is inscribed in a cone so that their centerlines coincide. The cone has height of 6 and radius of the base of 3. Find the dimensions of the cylinder that has maximum volume.

Let r be the radius of the base of the cylinder and let h be the height of the cylinder. Using similar triangles or otherwise we get:

$$\frac{3-r}{h} = \frac{3}{6}$$

which gives $h = 6 - 2r$. The volume of the cylinder is the given by the formula:

$$V = \pi r^2 h$$

$$V(r) = \pi r^2(6-2r) = 6\pi r^2 - 2\pi r^3$$

with $0 < r < 3$. Taking the derivative we get:

$$\frac{dV}{dr} = 12\pi r - 6\pi r^2$$

$$\frac{d^2V}{dr^2} = 12\pi - 12\pi r$$

Within the domain the derivative is 0 for $r = 2$ and the second derivative is positive at that value, so it is a maximum. If $r = 2$, then $h = 6 - 4 = 2$.

3. [5 *points*]

A balloon is rising vertically from a point on the ground that is 200 metres from an observer at ground level. The angle of elevation from the observer to the balloon is increasing at a rate of 9° per second at the instant this angle is 45°. How fast is the balloon rising at this time?

We have (remember to work in radians):

$$\frac{d\theta}{dt}_{|\theta=\frac{\pi}{4}} = \frac{\pi}{20}$$

We are looking for:

$$\frac{dh}{dt}_{|\theta=\frac{\pi}{4}} = ?$$

And the relationship between θ and h is as follows:

$$\tan\theta = \frac{h}{200}$$

Taking the derivative:

$$\sec^2\theta\frac{d\theta}{dt} = \frac{\frac{dh}{dt}}{200}$$

Which gives:

$$\frac{dh}{dt} = 200\sec^2\theta\frac{d\theta}{dt}$$

$$\frac{dh}{dt}_{|\theta=\frac{\pi}{4}} = 200\sec^2\left(\frac{\pi}{4}\right)\frac{\pi}{20} = 20\pi\ \frac{m}{s}$$

4. [5 *points*]

Water is evaporating from a cup in a shape of an inverted cone. The rate of evaporation is proportional to the area of the surface of the water. Show that the depth of the water decreases at a constant rate that does not depend on the dimensions of the cup.

Let r and h be the dimensions of the water and let R and H be the dimensions of the cone. Note that r and h are variables and R and H are constants. We have:

$$\frac{r}{h} = \frac{R}{H}$$

We also have:

$$\frac{dV}{dt} = -kr^2$$

where k is some positive constant.

$$V = \frac{1}{3}\pi r^2 h$$

$$V(h) = \frac{1}{3}\pi \frac{R^2}{H^2} h^3$$

Taking the derivative we get:

$$\frac{dV}{dt} = \pi \frac{R^2}{H^2} h^2 \frac{dh}{dt}$$

$$-kr^2 = \pi \frac{R^2}{H^2} h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{kr^2}{\pi \frac{R^2}{H^2} h^2} = -\frac{k}{\pi} \frac{r^2}{h^2} \frac{H^2}{R^2}$$

Note that $\frac{r^2}{h^2}\frac{H^2}{R^2} = 1$, so we get:

$$\frac{dh}{dt} = -\frac{k}{\pi}$$

which is a constant and does not depend on R or H.