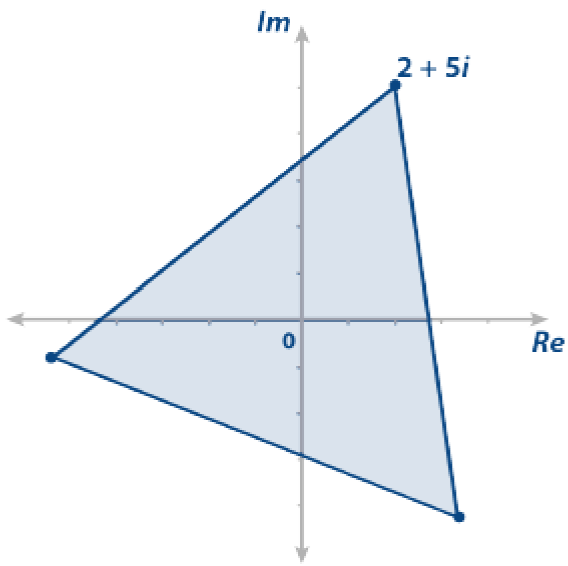
Complex numbers - revision [104 marks]

1. [Maximum mark: 5] EXN.1.AHL.TZ0.14

(a) Write down $2+5\mathrm{i}$ in exponential form. [2]

(b)



An equilateral triangle is to be drawn on the Argand plane with one of the vertices at the point corresponding to $2+5\mathrm{i}$ and all the vertices equidistant from 0.

Find the points that correspond to the other two vertices. Give your answers in Cartesian form. [3]

2. [Maximum mark: 6] 24M.1.AHL.TZ1.13

Let $z_1 = 4+5\mathrm{i}$.

(a.i) Find $|z_1|$. [1]

(a.ii) Find $\arg(z_1)$. [1]

Let $z_2 = 3\mathrm{e}^{2\mathrm{i}}$.

(b) Find the area of the triangle on an Argand diagram with vertices $0,\ z_1$ and z_2. [4]

3. [Maximum mark: 6] 23N.1.AHL.TZ0.8

Given $z = \sqrt{3} - \mathrm{i}$.

(a) Write z in the form $z = r\mathrm{e}^{\theta \mathrm{i}}$, where $r \in \mathbb{R}^+, -\pi < \theta \leq \pi$. [2]

Let $z_1 = \mathrm{e}^{2t\mathrm{i}}$ and $z_2 = 2\mathrm{e}^{\left(2t - \frac{\pi}{6}\right)\mathrm{i}}$.

(b) Find $\mathrm{Im}\ (z_1\ +\ z_2)$ in the form $p\ \sin\ (2t\ +\ q)$, where $p > 0, t \in \mathbb{R}$ and $-\pi \leq q \leq \pi$. [4]

4. [Maximum mark: 5] 23M.1.AHL.TZ1.12

Two AC (alternating current) electrical sources with the same frequencies are combined. The
voltages from these sources can be expressed as $V_1 = 6\ \sin\ (at + 30^\circ)$ and $V_2 = 6\ \sin\ (at + 90^\circ)$.

The combined total voltage can be expressed in the form $V_1 + V_2 = V\ \sin\ (at + \theta^\circ)$.

Determine the value of V and the value of θ. [5]

5. [Maximum mark: 6] 23M.1.AHL.TZ2.11

Two AC (alternating current) electrical sources of equal frequencies are combined.

The voltage of the first source is modelled by the equation $V = 30\ \sin\ (t + 60^\circ)$.

The voltage of the second source is modelled by the equation
$V = 60 \sin (t + 10^\circ)$.

(a) Determine the maximum voltage of the combined sources. [2]

(b) Using your graphic display calculator, find a suitable equation for the combined voltages, giving your answer in the form $V = V_0 \sin (at + b)$, where a, b and V_0 are constants, $a > 0$ and $0^\circ \leq b \leq 180^\circ$. [4]

6. [Maximum mark: 8] 22N.1.AHL.TZ0.17

The time of sunrise, R hours after midnight, in Taipei can be modelled by

$$R = 1.08 \cos(0.0165t + 0.413) + 4.94,$$

where t is the day of the year 2021 (for example, $t = 2$ represents 2 January 2021).

The time of sunset, S hours after midnight, in Taipei can be modelled by

$$S = 1.15 \cos(0.0165t - 2.97) + 18.9.$$

The number of daylight hours, D, in Taipei during 2021 can be modelled by

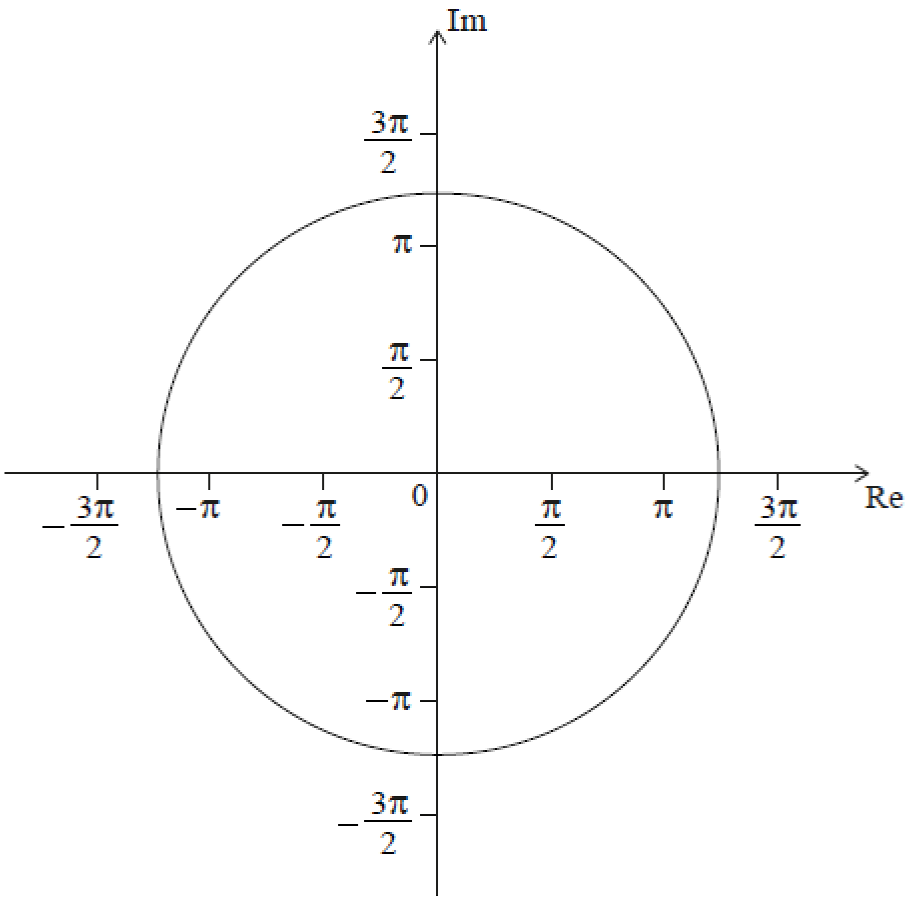
$$D = a \cos(0.0165t + b) + c.$$

(a) Find the value of a, of b and of c. [6]

(b) Hence, or otherwise, find the largest number of daylight hours in Taipei during 2021 and the day of the year on which this occurs. [2]

7. [Maximum mark: 7] 22M.1.AHL.TZ1.10

The following Argand diagram shows a circle centre 0 with a radius of 4 units.



A set of points, $\{z_\theta\}$, on the Argand plane are defined by the equation

$z_\theta = \frac{1}{2}\theta e^{\theta i}$, where $\theta \geq 0$.

Plot on the Argand diagram the points corresponding to

(a.i) $\theta = \frac{\pi}{2}$. [1]

(a.ii) $\theta = \pi$. [1]

(a.iii) $\theta = \frac{3\pi}{2}$. [1]

Consider the case where $|z_\theta| = 4$.

(b.i) Find this value of θ. [2]

(b.ii) For this value of θ, plot the approximate position of z_θ on the Argand diagram. [2]

8. [Maximum mark: 5] 22M.1.AHL.TZ2.13

An electric circuit has two power sources. The voltage, V_1, provided by the first power source, at time t, is modelled by

$$V_1 = \mathrm{Re}\left(2\mathrm{e}^{3t\mathrm{i}}\right).$$

The voltage, V_2, provided by the second power source is modelled by

$$V_2 = \mathrm{Re}\left(5\mathrm{e}^{(3t+4)\mathrm{i}}\right).$$

The total voltage in the circuit, V_T, is given by

$$V_T = V_1 + V_2.$$

(a) Find an expression for V_T in the form $A\cos(Bt + C)$, where $A,\ B$ and C are real constants. [4]

(b) Hence write down the maximum voltage in the circuit. [1]

9. [Maximum mark: 13] 21N.2.AHL.TZ0.5

Let $z = 1 - \mathrm{i}$.

(a.i) Plot the position of z on an Argand Diagram. [1]

(a.ii) Express z in the form $z = a\mathrm{e}^{\mathrm{i}b}$, where $a,\ b \in \mathbb{R}$, giving the exact value of a and the exact value of b. [2]

Let $w_1 = \mathrm{e}^{\mathrm{i}x}$ and $w_2 = \mathrm{e}^{\mathrm{i}\left(x - \frac{\pi}{2}\right)}$, where $x \in \mathbb{R}$.

(b.i) Find $w_1 + w_2$ in the form $\mathrm{e}^{\mathrm{i}x}(c + \mathrm{i}d)$. [2]

(b.ii) Hence find $\mathrm{Re}(w_1 + w_2)$ in the form $A\cos(x - a)$, where $A > 0$ and $0 < a \leq \frac{\pi}{2}$. [4]

The current, I, in an AC circuit can be modelled by the equation $I = a\cos(bt - c)$ where b is the frequency and c is the phase shift.

Two AC voltage sources of the same frequency are independently connected to the same circuit. If connected to the circuit alone they generate currents I_{A} and I_{B}. The maximum value and the phase shift of each current is shown in the following table.

Current	Maximum value	Phase shift
I_{A}	12 amps	0
I_{B}	12 amps	$\frac{\pi}{2}$

When the two voltage sources are connected to the circuit at the same time, the total current I_{T} can be expressed as $I_{\mathrm{A}} + I_{\mathrm{B}}$.

(c.i) Find the maximum value of I_{T}. [3]

(c.ii) Find the phase shift of I_{T}. [1]

10. [Maximum mark: 8] 21M.1.AHL.TZ1.9

Consider $w = \mathrm{i}z + 1$, where $w,\ z \in \mathbb{C}$.

Find w when

(a.i) $z = 2\mathrm{i}$. [2]

(a.ii) $z = 1 + \mathrm{i}$. [1]

Point z on the Argand diagram can be transformed to point w by two transformations.

(b) Describe these two transformations and give the order in which they are applied. [3]

(c) Hence, or otherwise, find the value of z when $w = 2 - \mathrm{i}$. [2]

11. [Maximum mark: 8] 21M.1.AHL.TZ2.12

It is given that $z_1 = 3\,\mathrm{cis}\left(\frac{3\pi}{4}\right)$ and $z_2 = 2\,\mathrm{cis}\left(\frac{n\pi}{16}\right),\ n \in \mathbb{Z}^+$.

In parts (a)(i) and (a)(ii), give your answers in the form $r\mathrm{e}^{\mathrm{i}\theta},\ r \geq 0,\ \ -\pi < \theta \leq \pi$.

(a.i) Find the value of ${z_1}^3$. [2]

(a.ii) Find the value of $\left(\frac{z_1}{z_2}\right)^4$ for $n = 2$. [3]

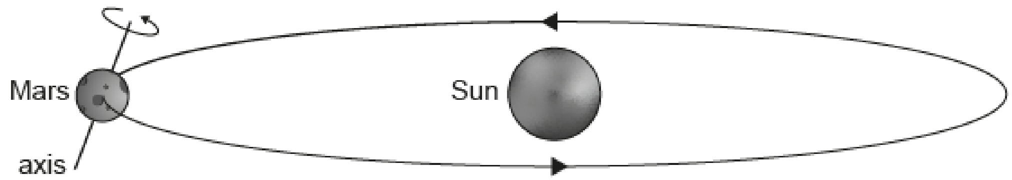
(b) Find the least value of n such that $z_1 z_2 \in \mathbb{R}^+$. [3]

12. [Maximum mark: 27] 21M.3.AHL.TZ1.1

A suitable site for the landing of a spacecraft on the planet Mars is identified at a point, $\mathbf{A}$. The shortest time from sunrise to sunset at point $\mathbf{A}$ must be found.

Radians should be used throughout this question. All values given in the question should be treated as exact.

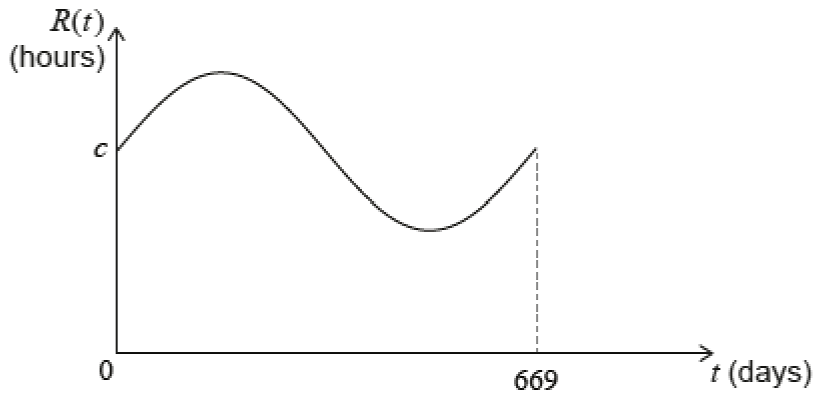
Mars completes a full orbit of the Sun in 669 Martian days, which is one Martian year.



On day t, where $t \in \mathbb{Z}$, the length of time, in hours, from the start of the Martian day until sunrise at point A can be modelled by a function, $R(t)$, where

$$R(t) = a\,\sin(bt) + c,\ t \in \mathbb{R}.$$

The graph of R is shown for one Martian year.



(a) Show that $b \approx 0.00939$. [2]

Mars completes a full rotation on its axis in 24 hours and 40 minutes.

(b) Find the angle through which Mars rotates on its axis each hour. [3]

The time of sunrise on Mars depends on the angle, δ, at which it tilts towards the Sun. During a Martian year, δ varies from -0.440 to 0.440 radians.

The angle, ω, through which Mars rotates on its axis from the start of a Martian day to the moment of sunrise, at point A, is given by $\cos\,\omega = 0.839\,\tan\,\delta$, $0 \le \omega \le \pi$.

(c.i) Show that the maximum value of $\omega = 1.98$, correct to three significant figures. [3]

(c.ii) Find the minimum value of ω. [1]

Use your answers to parts (b) and (c) to find

(d.i) the maximum value of $R(t)$. [2]

(d.ii) the minimum value of $R(t)$. [1]

(e) Hence show that $a = 1.6$, correct to two significant figures. [2]

(f) Find the value of c. [2]

Let $S(t)$ be the length of time, in hours, from the start of the Martian day until **sunset** at point A on day t. $S(t)$ can be modelled by the function

$$S(t) = 1.5\sin(0.00939t + 2.83) + 18.65.$$

The length of time between sunrise and sunset at point A, $L(t)$, can be modelled by the function

$$L(t) = 1.5\sin(0.00939t + 2.83) - 1.6\sin(0.00939t) + d.$$

(g) Find the value of d. [2]

Let $f(t) = 1.5\sin(0.00939t + 2.83) - 1.6\sin(0.00939t)$ and hence $L(t) = f(t) + d$.

$f(t)$ can be written in the form $\mathrm{Im}(z_1 - z_2)$, where z_1 and z_2 are complex functions of t.

(h.i) Write down z_1 and z_2 in exponential form, with a constant modulus. [3]

(h.ii) Hence or otherwise find an equation for L in the form $L(t) = p\sin(qt + r) + d$, where $p,\ q,\ r,\ d \in \mathbb{R}$. [4]

(h.iii) Find, in hours, the shortest time from sunrise to sunset at point A that is predicted by this model. [2]