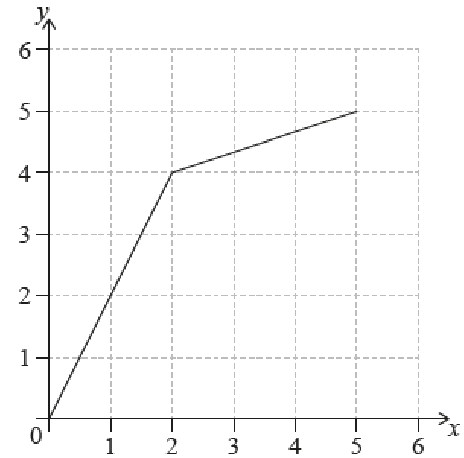
Functions revision [67 marks]

1. [Maximum mark: 7] 23N.1.AHL.TZ0.2

The graph of the function f is given in the following diagram.



(a) Write down $f(2)$. [1]

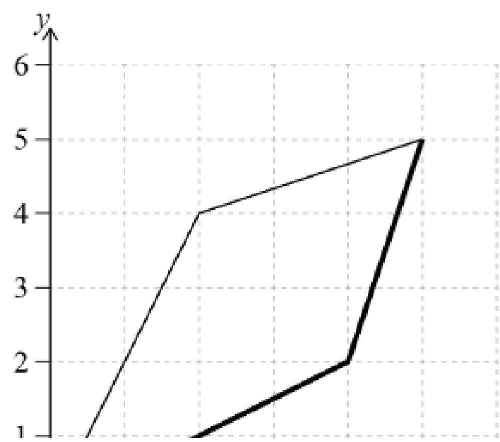
Markscheme

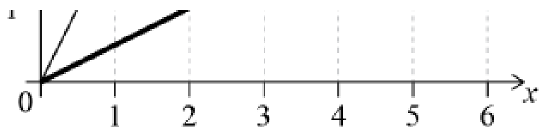
4 ***A1***

[1 mark]

(b) On the axes, sketch $y = f^{-1}(x)$. [2]

Markscheme





A1A1

Note: Award ***A1*** for passing through $(0,\ 0)$ and $(4,\ 2)$, ***A1*** for passing through $(4,\ 2)$ and $(5,\ 5)$.

[2 marks]

The function g is defined as $g(x) = 3x - 1$.

(c) Find an expression for $g^{-1}(x)$ [2]

Markscheme

attempt to solve $y = 3x - 1$ for x **OR** changing variables ***(M1)***

$\left(g^{-1}(x)\right) = \frac{x+1}{3}$ ***A1***

[2 marks]

(d) Find a value of x where $f^{-1}(x) = g^{-1}(x)$. [2]

Markscheme

sketch of $g(x)$ or $g^{-1}(x)$, algebraic approach ***(M1)***

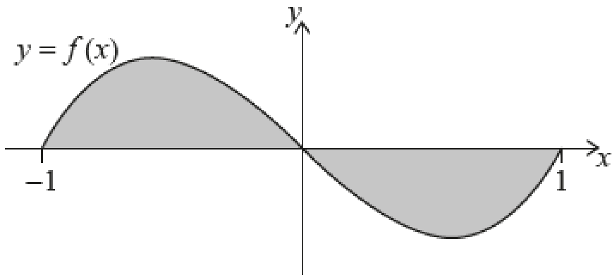
$\frac{1}{2}x = \frac{x+1}{3}$

$(x =)\ 2$ ***A1***

[2 marks]

2. [Maximum mark: 7] 23N.1.AHL.TZ0.11

Consider the function $f(x) = x^3 - x$, for $-1 \leq x \leq 1$. The shaded region, R, is bounded by the graph of $y = f(x)$ and the x-axis.



(a.i) Write down an integral that represents the area of R. [1]

Markscheme

EITHER

(area of $R =$) $\int_{-1}^{1}\left|x^3 - x\right| \mathrm{d}\, x$ ***A1***

OR

(area of $R =$) $2 \times \int_{-1}^{0} x^3 - x \mathrm{~d}\, x$ **OR** (area of $R =$) $-2 \times \int_{0}^{1} x^3 - x \mathrm{~d}\, x$ ***A1***

OR

(area of $R =$) $\int_{-1}^{0} x^3 - x \mathrm{~d}\, x - \int_{0}^{1} x^3 - x \mathrm{~d}\, x$ ***A1***

[1 mark]

(a.ii) Find the area of R. [1]

Markscheme

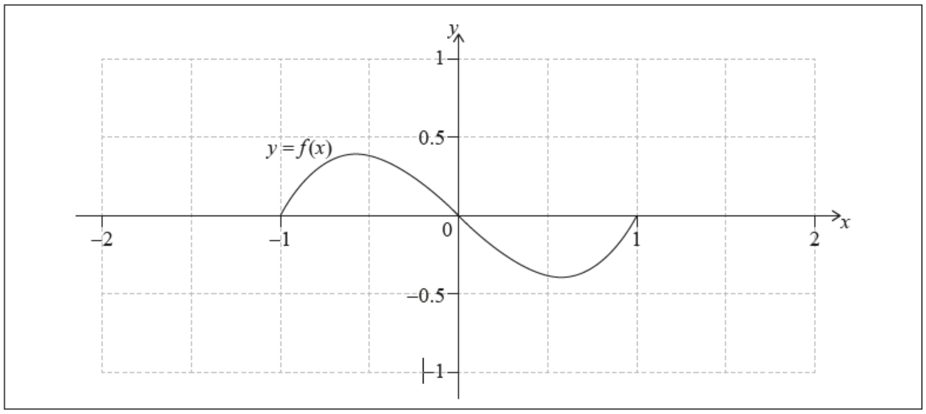
(area of $R =$) 0.5 ***A1***

Note: Follow through from part (a)(i) only if answer is greater than zero.

[1 mark]

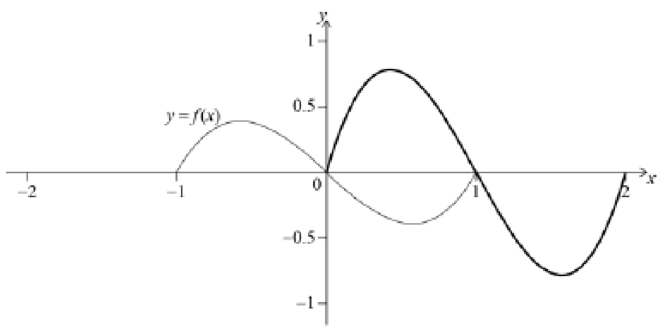
Another function, g, is defined such that $g(x) = 2f(x - 1)$.

(b) On the following set of axes, the graph of $y = f(x)$ has been drawn. On the same set of axes, sketch the graph of $y = g(x)$.



[2]

Markscheme



A1A1

Note: Award ***A1*** for sketch with correct shape on $[0,\ 2]$, ***A1*** for vertical stretch x2. Condone max/min of g extending to 1 / -1.

[2 marks]

The region R from the original graph $y = f(x)$ is rotated through 2π radians about the x-axis to form a solid.

(c) Find the volume of the solid. [3]

Markscheme

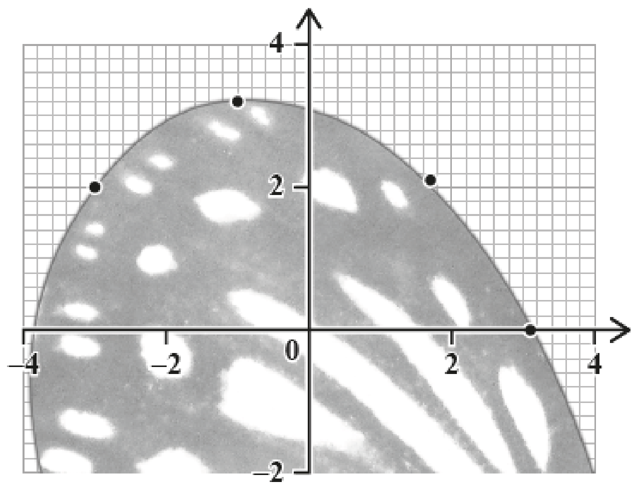
attempt to use $\pi \int y^2\ \mathrm{d}\ x$ ***(M1)***

volume $= \pi \int_{-1}^{1} \left(\mathrm{x}^3 - x\right)^2\ \mathrm{d}\ x$ ***(A1)*** volume $= 0.479$ (cubic units) $\left(= 0.478718\ldots,\ \frac{16\pi}{105}\right)$ ***A1***

[3 marks]

3. [Maximum mark: 5] 22N.1.AHL.TZ0.11

Gloria wants to model the curved edge of a butterfly wing. She inserts a photo of the wing into her graphing software and finds the coordinates of four points on the edge of the wing.



x	y
–3	2
–1	3.2
1.7	2.1
3.1	0

Gloria thinks a cubic curve will be a good model for the butterfly wing.

[Source: Fleur, 2019. photo-1560263816-d704d83cce0f. [image online] Available at: https://unsplash.com/photos/SE2zTdS1MNo [Accessed 8 February 2022]. Source adapted.]

(a) Find the equation of the cubic regression curve for this data. [2]

Markscheme

$y = -0.00855x^3 - 0.234x^2 - 0.225x + 3.20$ ***A2***

$\left(y = -0.00854819\ldots x^3 - 0.234002\ldots x^2 - 0.224884\ldots x + 3.20056\ldots\right)$

Note: Award ***A0A1*** for at least two terms correct.

[2 marks]

For the photo of a second butterfly wing, Gloria finds the equation of the regression curve is $y = 0.0083x^3 - 0.075x^2 - 0.58x + 2.2$.

Gloria realizes that her photo of the second butterfly is an enlargement of the life-size butterfly, scale factor 2 and centred on $(0,\ 0)$.

(b) Find the equation of the cubic curve that models the life-size wing. [3]

Markscheme

$y(2x)$ (for horizontal stretch) ***(A1)***

attempt to stretch vertically by factor $\frac{1}{2}$ ***(M1)***

$y = 0.0332x^3 - 0.15x^2 - 0.58x\,(+1.1)$ ***A1***

Note: Award ***A0M1A0*** for a vertical stretch, factor 2. Although a d value of 1.1 is preferred, technically this value can be wrong/omitted and the question is still answered (hence it is presented in brackets).

[3 marks]

4. [Maximum mark: 5] 22M.1.AHL.TZ2.10

The function $f(x) = \ln\left(\frac{1}{x-2}\right)$ is defined for $x > 2,\ x \in \mathbb{R}$.

(a) Find an expression for $f^{-1}(x)$. You are not required to state a domain. [3]

Markscheme

$y = \ln\left(\frac{1}{x-2}\right)$

an attempt to isolate x (or y if switched) ***(M1)***

$e^y = \frac{1}{x-2}$

$x-2=\mathrm{e}^{-y}$

$x=\mathrm{e}^{-y}+2$

switching x and y (seen anywhere) ***M1***

$f^{-1}(x)=\mathrm{e}^{-x}+2$ ***A1***

[3 marks]

(b) Solve $f(x)=f^{-1}(x)$. [2]

Markscheme

sketch of $f(x)$ and $f^{-1}(x)$ ***(M1)***

$x=2.12\ \ (2.12002\ldots)$ ***A1***

[2 marks]

5. [Maximum mark: 18] 22M.2.AHL.TZ1.6

Consider the curve $y=\sqrt{x}$.

(a.i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

Markscheme

$y=x^{\frac{1}{2}}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}x^{-\frac{1}{2}}$ ***A1***

[2 marks]

(a.ii) Hence show that the equation of the tangent to the curve at the point $(0.16,\ 0.4)$ is $y=1.25x+0.2$. [2]

Markscheme

gradient at $x = 0.16$ is $\frac{1}{2} \times \frac{1}{\sqrt{0.16}}$ ***M1***

$= 1.25$

EITHER

$y - 0.4 = 1.25(x - 0.16)$ ***M1***

OR

$0.4 = 1.25(0.16) + b$ ***M1***

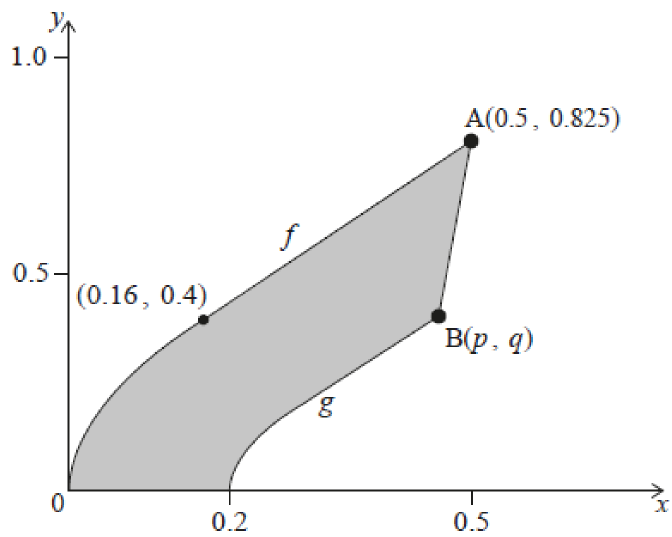
Note: Do not allow working backwards from the given answer.

THEN

hence $y = 1.25x + 0.2$ ***AG***

[2 marks]

The shape of a piece of metal can be modelled by the region bounded by the functions f, g, the x-axis and the line segment $[\mathrm{AB}]$, as shown in the following diagram. The units on the x and y axes are measured in metres.



The piecewise function f is defined by

$$f(x) = \begin{cases} \sqrt{x} & 0 \leq x \leq 0.16 \\ 1.25x + 0.2 & 0.16 < x \leq 0.5 \end{cases}$$

The graph of g is obtained from the graph of f by:

- a stretch scale factor of $\frac{1}{2}$ in the x direction,
- followed by a stretch scale factor $\frac{1}{2}$ in the y direction,
- followed by a translation of 0.2 units to the right.

Point A lies on the graph of f and has coordinates $(0.5,\ 0.825)$. Point B is the image of A under the given transformations and has coordinates $(p,\ q)$.

(b) Find the value of p and the value of q. [2]

Markscheme

$p = 0.45, \quad q = 0.4125$ (or 0.413) (accept " $(0.45,\ 0.4125)$ ") ***A1A1***

[2 marks]

The piecewise function g is given by

$$g(x) = \begin{cases} h(x) & 0.2 \leq x \leq a \\ 1.25x + b & a < x \leq p \end{cases}$$

(c.i) Find an expression for $h(x)$.

[2]

Markscheme

$(h(x) =)\ \frac{1}{2}\sqrt{2(x-0.2)}$ ***A2***

Note: Award ***A1*** if only two correct transformations are seen.

[2 marks]

(c.ii) Find the value of a. [1]

Markscheme

$(a =)\ 0.28$ ***A1***

[1 mark]

(c.iii) Find the value of b. [2]

Markscheme

EITHER

Correct substitution of their part (b) (or $(0.28,\ 0.2)$) into the given expression ***(M1)***

OR

$\frac{1}{2}(1.25 \times 2(x-0.2)+0.2)$ ***(M1)***

Note: Award ***M1*** for transforming the equivalent expression for f correctly.

THEN

$(b =)\ -0.15$ ***A1***

[2 marks]

(d.i) Find the area enclosed by $y = f(x)$, the x-axis and the line $x = 0.5$. [3]

Markscheme

recognizing need to add two integrals ***(M1)***

$\int_0^{0.16} \sqrt{x}\ \mathrm{d}\ x + \int_{0.16}^{0.5} (1.25x + 0.2)\ \mathrm{d}\ x$ ***(A1)***

Note: The second integral could be replaced by the formula for the area of a trapezoid $\frac{1}{2} \times 0.34(0.4 + 0.825)$.

$0.251\,\mathrm{m}^2\ \ (0.250916\ldots)$ ***A1***

[3 marks]

The area enclosed by $y = g(x)$, the x-axis and the line $x = p$ is $0.0627292\,\mathrm{m}^2$ correct to six significant figures.

(d.ii) Find the area of the shaded region on the diagram. [4]

Markscheme

EITHER

area of a trapezoid $\frac{1}{2} \times 0.05(0.4125 + 0.825) = 0.0309375$ ***(M1)(A1)***

OR

$\int_{0.45}^{0.5} (8.25x - 3.3)\ \mathrm{d}\ x = 0.0309375$ ***(M1)(A1)***

Note: If the rounded answer of 0.413 from part (b) is used, the integral is $\int_{0.45}^{0.5} (8.24x - 3.295)\ \mathrm{d}\ x = 0.03095$ which would be awarded ***(M1)(A1)***.

THEN

shaded area $= 0.250916\ldots - 0.0627292 - 0.0309375$ ***(M1)***

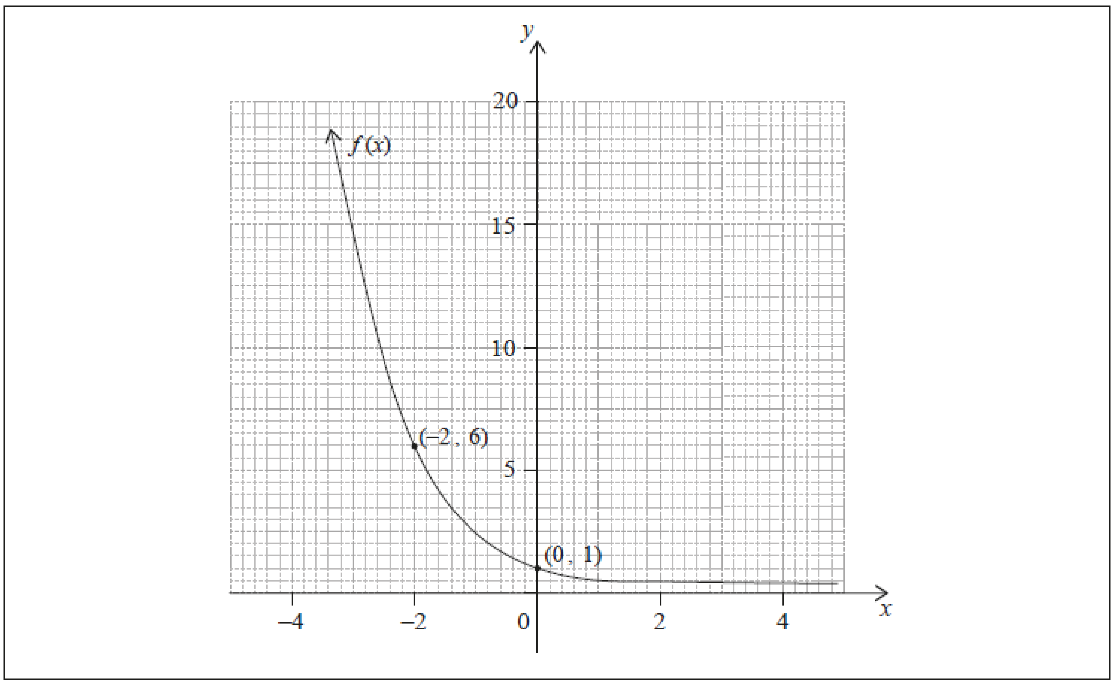
Note: Award ***(M1)*** for the subtraction of both $0.0627292\ldots$ and their area for the trapezoid from their answer to (a)(i).

$= 0.157\,\text{m}^2 \;\; (0.15725)$ ***A1***

[4 marks]

6. [Maximum mark: 4] 21N.1.AHL.TZ0.10

The graph of $y = f(x)$ is given on the following set of axes. The graph passes through the points $(-2,\ 6)$ and $(0,\ 1)$, and has a horizontal asymptote at $y = 0$.



Let $g(x) = 2f(x-2) + 4$.

(a) Find $g(0)$. [2]

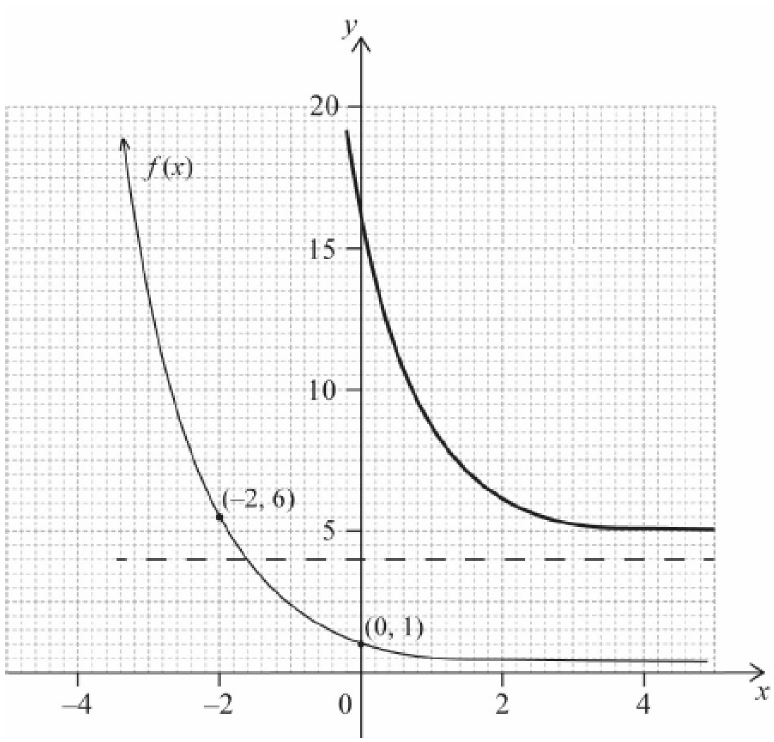
Markscheme

$g(0) = 16$ ***M1A1***

[2 marks]

(b) On the same set of axes draw the graph of $y = g(x)$, showing any intercepts and asymptotes. [2]

Markscheme



y-asymptote $(y = 4)$ ***A1***

concave up decreasing curve and passing through $(0,\ 16)$ ***A1***

[2 marks]

7. [Maximum mark: 7] 21M.1.AHL.TZ1.17

The graph of the function $f(x) = \ln x$ is translated by $\begin{pmatrix} a \\ b \end{pmatrix}$ so that it then passes through the points $(0,\ 1)$ and $\left(\mathrm{e}^3,\ 1 + \ln 2\right)$.

Find the value of a and the value of b. [7]

Markscheme

new function is $f(x-a)+b(=\ln(x-a)+b)$ ***(M1)***

$f(0) = \ln(-a) + b = 1$ ***A1***

$f\left(\mathrm{e}^3\right) = \ln\left(\mathrm{e}^3 - a\right) + b = 1 + \ln 2$ ***A1***

$\ln(-a) = \ln\left(\mathrm{e}^3 - a\right) - \ln 2$ ***(M1)***

$\ln(-a) = \ln\left(\frac{\mathrm{e}^3 - a}{2}\right)$

$-a = \frac{\mathrm{e}^3 - a}{2}$

$-2a = \mathrm{e}^3 - a$

$a = -\mathrm{e}^3\ \ (= -20.0855\ldots)$ ***A1***

$b = 1 - \ln \mathrm{e}^3 = 1 - 3 = -2$ ***(M1)A1***

[7 marks]

8. [Maximum mark: 7] 21M.1.AHL.TZ2.2

A function is defined by $f(x) = 2 - \frac{12}{x+5}$ for $-7 \le x \le 7,\ x \ne -5$.

(a) Find the range of f. [3]

Markscheme

$(f(-7) =)\ 8$ and $(f(7) =)\ 1$ ***(A1)***

range is $f(x) \le 1,\ f(x) \ge 8$ ***A1A1***

Note: Award at most ***A1A1A0*** if strict inequalities are used.

[3 marks]

(b) Find an expression for the inverse function $f^{-1}(x)$. The domain is not required. [3]

Markscheme

interchanging $x,\ y$ at any stage ***(A1)***

$y = 2 - \frac{12}{x+5}$

$\frac{12}{x+5} = 2 - y$

$\frac{12}{2-y} = x + 5$ ***(A1)***

$\frac{12}{2-y} - 5 = x$

$\left(f^{-1}(x) =\right) \frac{12}{2-x} - 5 \ \left(= \frac{2+5x}{2-x}\right)$ ***A1***

[3 marks]

(c) Write down the range of $f^{-1}(x)$. [1]

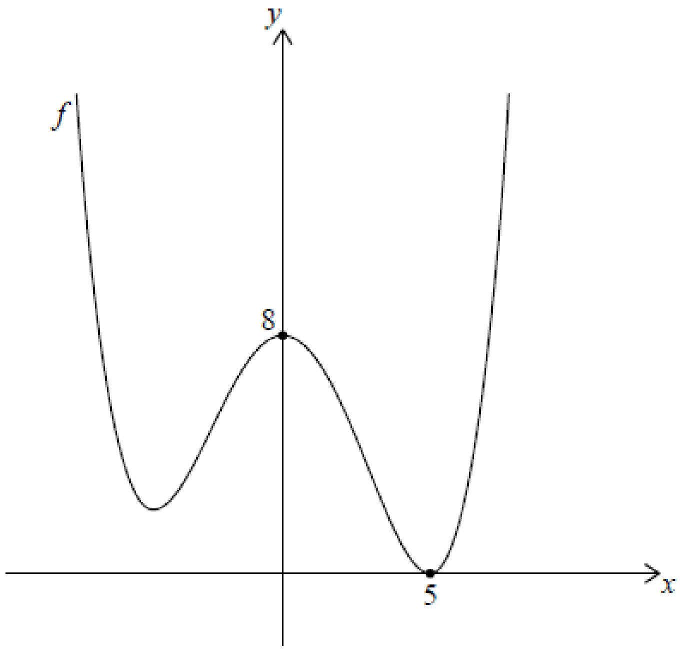
Markscheme

range is $-7 \leq f^{-1}(x) \leq 7,\ \ f^{-1}(x) \neq -5$ ***A1***

[1 mark]

9. [Maximum mark: 7] 19M.1.SL.TZ2.S_4

The following diagram shows part of the graph of f with x-intercept (5, 0) and y-intercept (0, 8).



(a.i) Find the y-intercept of the graph of $f(x)+3$. [1]

Markscheme

y-intercept is 11 (accept (0, 11)) ***A1 N1***

[1 mark]

(a.ii) Find the y-intercept of the graph of $f(4x)$. [2]

Markscheme

valid approach ***(M1)***

eg $f(4\times 0)=f(0)$, recognizing stretch of $\frac{1}{4}$ in x-direction

y-intercept is 8 (accept (0, 8)) ***A1 N2***

[2 marks]

(b) Find the x-intercept of the graph of $f(2x)$. [2]

Markscheme

x-intercept is $\frac{5}{2}$ $(=2.5)$ (accept $\left(\frac{5}{2},\ 0\right)$ or (2.5, 0)) ***A2 N2***

[2 marks]

(c) Describe the transformation $f\left(x+1\right)$. [2]

Markscheme

correct name, correct magnitude **and** direction ***A1A1 N2***

eg name: translation, (horizontal) shift (do not accept move)

eg magnitude and direction: 1 unit to the left, $\begin{pmatrix}-1\\0\end{pmatrix}$, horizontal by –1

[2 marks]