Logarithms - revision [81 marks]

1. [Maximum mark: 6] EXN.1.SL.TZ0.5

The pH of a solution is given by the formula $pH = -\log_{10} C$ where C is the hydrogen ion concentration in a solution, measured in moles per litre (Ml^{-1}).

(a) Find the pH value for a solution in which the hydrogen ion concentration is 5.2×10^{-8}. [2]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

$pH = -\log_{10}\left(5.2 \times 10^{-8}\right) = 7.29\ (7.28399\ldots)$ **(M1)A1**

[2 marks]

(b.i) Write an expression for C in terms of pH. [2]

Markscheme

$C = 10^{-pH}$ **(M1)A1**

Note: Award **M1** for an exponential equation with 10 as the base.

[2 marks]

(b.ii) Find the hydrogen ion concentration in a solution with $pH\ 4.2$. Give your answer in the form $a \times 10k$ where $1 \leq a < 10$ and k is an integer. [2]

Markscheme

$C = 10^{-4.2} = 6.30957\ldots \times 10^{-5}$ **(M1)**

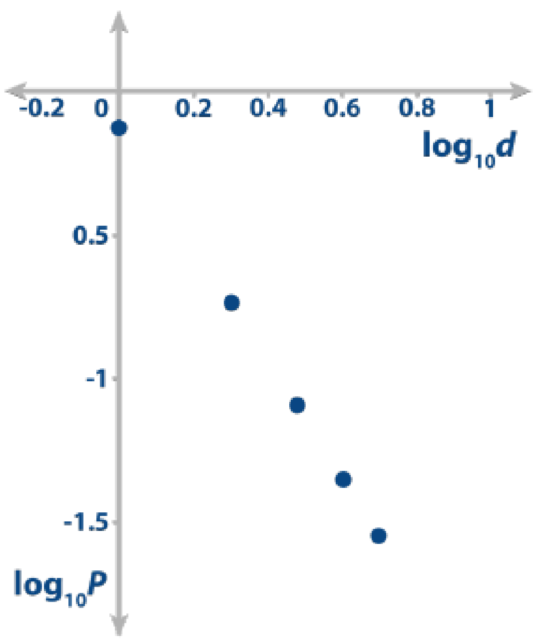
6.31×10^{-5} **A1**

[2 marks]

2. [Maximum mark: 7] EXN.1.AHL.TZ0.12

It is believed that the power P of a signal at a point d km from an antenna is inversely proportional to d^n where $n \in \mathbb{Z}^+$.

The value of P is recorded at distances of $1\,\text{m}$ to $5\,\text{m}$ and the values of $\log_{10} d$ and $\log_{10} P$ are plotted on the graph below.



(a) Explain why this graph indicates that P is inversely proportional to d^n.

[2]

Markscheme

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a straight line with a negative gradient **A1A1**

[2 marks]

The values of $\log_{10} d$ and $\log_{10} P$ are shown in the table below.

$\log_{10} d$	0	0.301	0.477	0.602	0.699
$\log_{10} P$	-0.127	-0.740	-1.10	-1.36	-1.55

(b) Find the equation of the least squares regression line of $\log_{10} P$ against $\log_{10} d$.

[2]

Markscheme

$\log P = -2.040\ldots \log d - 0.12632\ldots \approx -2.04 \log d - 0.126$ **A1A1**

Note: A1 for each correct term.

[2 marks]

(c.i) Use your answer to part (b) to write down the value of n to the nearest integer. [1]

Markscheme

$n = 2$ **A1**

[1 mark]

(c.ii) Find an expression for P in terms of d. [2]

Markscheme

$P = 10^{-0.126\ldots} d^{-2}$ **(M1)**

$\approx 0.748 d^{-2}$ **A1**

[2 marks]

3. [Maximum mark: 7] EXM.1.AHL.TZ0.14

It is believed that two variables, v and w are related by the equation $v = kw^n$, where $k,\ n \in \mathbb{R}$. Experimental values of v and w are obtained. A graph of $\ln v$ against $\ln w$ shows a straight line passing through (−1.7, 4.3) and (7.1, 17.5).

Find the value of k and of n. [7]

Markscheme

$\ln v = n \ln w + \ln k$ ***M1A1***

gradient $= \frac{17.5-4.3}{7.1+1.7}$ $(= 1.5)$ ***M1***

$n = 1.5$ ***A1***

y-intercept $= 1.5 \times 1.7 + 4.3\ (= 6.85)$ ***M1***

$k = e^{6.85} = 944$ ***M1A1***

[7 marks]

4. [Maximum mark: 10] EXM.1.AHL.TZ0.15

Adesh wants to model the cooling of a metal rod. He heats the rod and records its temperature as it cools.

Time, t (seconds)	0	30	60	90	120	150
Temperature, T (°C)	75.6	62.2	53.3	47.4	42.3	38.5

He believes the temperature can be modeled by $T(t) = a\mathrm{e}^{bt} + 25$, where $a,\ b \in \mathbb{R}$.

(a) Show that $\ln(T - 25) = bt + \ln a$. [2]

Markscheme

$\ln(T - 25) = \ln\left(a\mathrm{e}^{bt}\right)$ ***M1***

$\ln(T - 25) = \ln a + \ln\left(\mathrm{e}^{bt}\right)$ ***A1***

$\ln(T - 25) = bt + \ln a$ ***AG***

[2 marks]

(b) Find the equation of the regression line of $\ln(T - 25)$ on t. [3]

Markscheme

$\ln(T - 25) = -0.00870t + 3.89$ ***M1A1A1***

[3 marks]

Hence

(c.i) find the value of a and of b. [3]

Markscheme

$b = -0.00870$ ***A1***

$a = e^{3.89\ldots} = 49.1$ ***M1A1***

[3 marks]

(c.ii) predict the temperature of the metal rod after 3 minutes. [2]

Markscheme

$T(180) = 49.1e^{-0.00870(180)} + 25 = 35.2$ ***M1A1***

[2 marks]

5. [Maximum mark: 5] EXM.1.AHL.TZ0.13

It is believed that two variables, m and p are related. Experimental values of m and p are obtained. A graph of $\ln m$ against p shows a straight line passing through (2.1, 7.3) and (5.6, 2.4).

(a) Find the equation of the straight line, giving your answer in the form $\ln m = ap + b$, where $a,\ b \in \mathbb{R}$. [3]

Markscheme

gradient = −1.4 ***A1***

$\ln m - 7.3 = -1.4(p - 2.1)$ ***M1***

$\ln m = -1.4p + 10.24$ ***A1***

[3 marks]

Hence, find

(b.i) a formula for m in terms of p. [1]

Markscheme

$m = e^{-1.4p+10.24} \quad \left(= 28000e^{-1.4p}\right)$ ***A1***

[1 mark]

(b.ii) the value of m when $p = 0$. [1]

Markscheme

28000 ***A1***

[1 mark]

6. [Maximum mark: 7] 24M.1.SL.TZ2.7

The pH scale is a measure of the acidity of a solution. Its value is given by the formula

pH$= -\log_{10}\left[\mathrm{H}^{+}\right]$,

where $\left[\mathrm{H}^{+}\right]$ is the concentration of hydrogen ions in the solution (measured in moles per litre).

(a) Calculate the pH value if the concentration of hydrogen ions is 0.0003. [2]

Markscheme

correct substitution of 0.0003 into the formula ***(A1)***

$\mathrm{pH} = -\log_{10}(0.0003)$

$= 3.52 \quad (3.52287\ldots)$ ***A1***

[2 marks]

The pH of milk is 6.6.

(b) Calculate the concentration of hydrogen ions in milk. [2]

Markscheme

EITHER

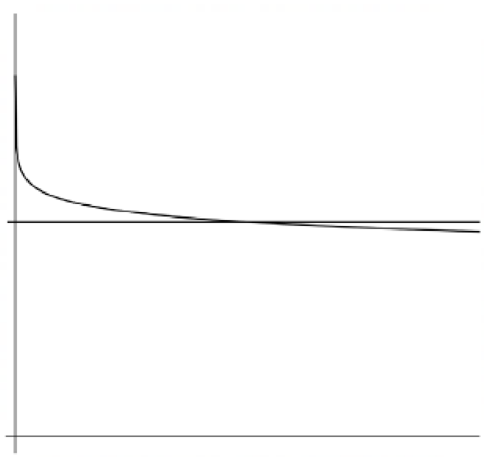
attempt to change to exponential form ***(M1)***

$[\mathrm{H}^+] = 10^{-6.6}$

OR

attempt to solve $6.6 = -\log_{10}[\mathrm{H}^+]$

(graphically or using numerical solver)



THEN

$[\mathrm{H}^+] = 0.000000251 \text{ (moles per litre)}$
$(0.000000251188\ldots,\ 2.51 \times 10^{-7})$ ***A1***

Note: Award ***M1A0*** for an answer of $2.51\ (2.51188\ldots)$ seen.

Award ***M0A0*** if a substitution of 6.6 into the formula is seen without an answer or some indication of using numerical solver.

[2 marks]

The strength of an acid is measured by its concentration of hydrogen ions.

A lemon has a pH value of 2 and a tomato has a pH value of 4.5.

(c) Calculate how many times stronger the acid in a lemon is when compared to the acid in a tomato. [3]

Markscheme

$2 = -\log_{10}[\mathrm{H}^+], 4.5 = -\log_{10}[\mathrm{H}^+]$

$10^{-2}\quad(0.01)$ **OR** $10^{-4.5}\quad(0.0000316227\ldots)$ ***(A1)***

substitution of their values into correct ratio

$\frac{10^{-2}}{10^{-4.5}}$ **OR** $\frac{0.01}{0.0000316227\ldots}$

$= 316.227\ldots = 316$ ***A1***

Note: Some candidates may subtract logs and hence look to solve $\log_{10}[\mathrm{H}^+] = 2.5$.

[3 marks]

7. [Maximum mark: 7] 24M.1.AHL.TZ1.9

(a) Find $\int \frac{8}{2x+3}\mathrm{d}x$. [3]

Markscheme

attempt to integrate by substitution or inspection ***(M1)***

$4 \ln |2x+3| + c$ **OR** $4 \ln |x+1.5| + c$ ***A1A1***

Note: Award ***M1*** for $\ln (2x+3)$ or $\ln (x+1.5)$, ***A1*** for the 4 and ***A1*** for c. The ***A*** marks can only be awarded if the ***M*** mark is awarded. Condone absence of modulus signs.

[3 marks]

(b) Hence find the exact area between the curve $y = \frac{8}{2x+3}$, the x-axis and the lines $x = 0$ and $x = 6$. Give your answer in the form $a \ln b$, where $a,\ b \in \mathbb{N}$. [4]

Markscheme

recognizing that area is $[4 \ln (2x+3)]_0^6$ ***(M1)***

$= 4 \ln (15) - 4 \ln (3)$ ***(A1)***

use of log laws for their expression ***(M1)***

$= 4 \ln (5)\ (= 2 \ln (25) = 1 \ln (625))$ ***A1***

Note: Award ***(M1)A0M0A0*** for an unsupported final answer of $6.43775\ldots$

Award at most ***(M1)A1FTM0A0*** if their answer from part (a) does not include $\ln$.

[4 marks]

8. [Maximum mark: 7] 23N.1.AHL.TZ0.10

The decay of a chemical isotope over five years is recorded in **Table 1**. The mass of the chemical M is measured to the nearest gram at the beginning of each year t of the experiment.

Table 1

Time t (years)	1	2	3	4	5
Mass M (grams)	1000	660	517	435	381

It is believed that the decay of the isotope can be modelled by an equation of the form $M = a \times t^b$.

(a) Use power regression on your graphic display calculator to find the value of a and the value of b. [2]

Markscheme

$M = 1000 \times t^{-0.6}$

$a = 1000 (= 999.972...) \quad b = -0.600 \; (-0.599991\ldots)$

A1A1

[2 marks]

The values of t and M can be transformed such that $x = \ln t$ and $y = \ln M$. **Table 2** shows data for x and y to three decimal places.

Table 2

x	0	0.693	1.099	1.386	1.609
y	6.908	6.492	6.248	6.075	5.943

(b) Find the linear regression equation of y on x, in the form $y = cx + d$. Give the values of c and d to three decimal places. [2]

Markscheme

$y = -0.600x + 6.908$

$c = -0.600 \qquad d = 6.908$ *A1A1*

Note: Long answer for c is $0.599991\ldots$ and for d is $6.90772\ldots$ If both answers are correct but not given to 3 decimal places award ***A1A0***.

[2 marks]

(c) Hence, show that this linear regression is equivalent to the power regression found in part (a). [3]

Markscheme

METHOD 1 (starting with the result in part (b))

attempt to apply addition (or subtraction) log laws ***M1***

attempt to apply inverse log ***M1***

Note: These ***M1*** marks can be applied in either order depending on the approach.

e.g. $\ln M = \ln t^{-0.600} + \ln \mathrm{e}^{6.908}$ then
$\ln M = \ln\left(\mathrm{e}^{6.908} \times t^{-0.600}\right)$

OR $\ln \frac{M}{t^{-0.600}} = 6.908$ then $\frac{M}{t^{-0.600}} = \mathrm{e}^{6.908}$

$M = \mathrm{e}^{6.908} t^{-0.600}$ ***A1***

$\left(M = 1000.24t^{-0.6}\right)$

$M = 1000t^{-0.6}$ and hence (close enough to be) equivalent ***AG***

Note: The ***AG*** line (or something which approximates it) must be seen for the final ***A1*** to be awarded. If 3 sf answers are used from part (b), the coefficient is 1002; this can be condoned in the working, as it equals 1000 when rounded to 3 sf.

METHOD 2 (starting with the result in part (a))

attempt to apply log ***M1***

$\ln M = \ln\left(1000 \times t^{-0.6}\right)$

attempt to apply addition (or subtraction) log laws ***M1***

$\ln M = \ln 1000 + \ln t^{-0.6}$

$\ln M = 6.90775\ldots - 0.6 \ln t$ ***A1***

$(\ln M = y, \ln t = x)$

$y = -0.600x + 6.908$ and hence (close enough to be) equivalent ***AG***

Note: The ***AG*** line (or something which approximates it) must be seen for the final ***A1*** to be awarded. Condone $b = -0.6$.

[3 marks]

9. [Maximum mark: 7] 23M.1.SL.TZ1.8

"Password entropy" is a measure of the predictability of a computer password. The higher the entropy, the more difficult it is to guess the password.

The relationship between the password entropy, p, (measured in bits) and the number of guesses, G, required to decode the password is given by $0.301p = \log_{10} G$.

(a) Calculate the value of p for a password that takes 5000 guesses to decode. [2]

Markscheme

attempt to substitute 5000 for G ***(M1)***

$0.301p = \log_{10} 5000$

$(p =)\ 12.3$ (bits) $(12.2889\ldots)$ ***A1***

[2 marks]

(b) Write down G as a function of p. [1]

Markscheme

$(G =) 10^{0.301p}$ **OR** 2^p ***A1***

[1 mark]

(c) Find the number of guesses required to decode a password that has an entropy of 28 bits. Write your answer in the form $a \times 10^k$, where $1 \le a < 10, k \in \mathbb{Z}$. [3]

Markscheme

attempt to substitute 28 for p in given equation or $G(p)$ ***(M1)***

$0.301 \times 28 = \log_{10} G$ **OR** $(G =)\ 10^{0.301 \times 28}$

$(G =)\ 2.68 \times 10^8\ (2.67916\ldots \times 10^8)$ ***A1A1***

Note: Award ***A1*** for 2.68, ***A1*** for 10^8. Award ***M1A1A0*** for a correct final answer not written in scientific notation or written incorrectly in scientific notation (e.g., $268\,000\,000$ or 26.8×10^7 or $2.68\text{E}08$).

[3 marks]

There is a point on the graph of the function $G(p)$ with coordinates $(0,\ 1)$.

(d) Explain what these coordinate values mean in the context of computer passwords. [1]

Markscheme

If a password has an **entropy of 0** (bits), then the password can be **guessed in one try** / then the **password is known** ***R1***

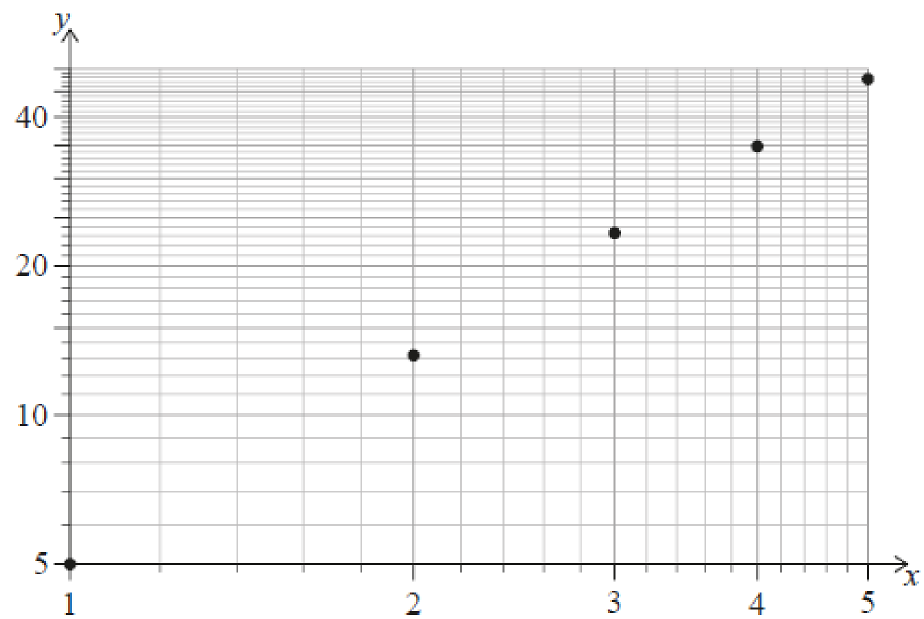
Note: Reference must be made to both entropy and number of guesses/password known for ***R1*** to be awarded.

Do not accept "no password" as this contradicts the context.

[1 mark]

10. [Maximum mark: 6] 23M.1.AHL.TZ1.14

Petra examines two quantities, x and y, and plots data points on a log-log graph.



She observes that on this graph the data points follow a perfect straight line. Given that the line passes through the points $(2,\ 13.1951)$ and $(4,\ 34.822)$, find the equation of the relationship connecting x and y. Your final answer should not include logarithms.

[6]

Markscheme

METHOD 1 Analytical approach

recognizing that the linear equation must be expressed in log form ***(M1)***

$\log y = m \log x + \log c$ (or $\log y = m \log x + C$)

EITHER

use of slope formula (must involve logs) ***(M1)***

$m = \frac{\log(34.822) - \log(13.1951)}{\log(4) - \log(2)} = 1.4$ ***A1***

attempt to substitute a value ***(M1)***

$\log c = \log(13.1951) - 1.4 \log 2 \; (= 0.69897\ldots)$

$\Rightarrow c = 5$ ***A1***

OR

$y = c.x^m$ ***(A1)***

attempt to set up two equations involving power functions ***(M1)***

$13.1951 = c \times 2^m$ and $34.822 = c \times 4^m$

$2^m = \frac{34.822}{13.1951} = 2.639\ldots \Rightarrow m = \log_2 2.639\ldots = 1.4$ ***A1***

$c = \frac{13.1951}{2.639\ldots} = 5$ ***A1***

THEN

(so the equation is) $y = 5 \times x^{1.4}$ ***A1***

METHOD 2 Regression analysis

recognizing that a log-log graph results in a power function model ***(M1)***

$y = a \times x^b$

attempt to find a power regression model using the given two points ***(M1)***

$a = 5$ and $b = 1.4$ ***(A1)(A1)***

(so the equation is) $y = 5 \times x^{1.4}$ ***A2***

[6 marks]

11. [Maximum mark: 6] 22N.1.SL.TZ0.10

Stars are classified by their brightness. The brightest stars in the sky have a magnitude of 1. The magnitude, m, of another star can be modelled as a function of its brightness, b, relative to a star of magnitude 1, as shown by the following equation.

$$m = 1 - 2.5 \log_{10}(b)$$

The star called Acubens has a brightness of 0.0525.

(a) Find the magnitude of Acubens. [2]

Markscheme

$m = 1 - 2.5 \log_{10}(0.0525)$ ***(M1)***

$= 4.20 \ \ (4.19960\ldots)$ ***A1***

[2 marks]

Ceres has a magnitude of 7 and is the least bright star visible without magnification.

(b) Find the brightness of Ceres. [2]

Markscheme

attempt to solve $7 = 1 - 2.5 \log_{10}(b)$ ***(M1)***

Note: Accept a sketch from their GDC as an attempt to solve $7 = 1 - 2.5 \log_{10}(b)$.

$b = 0.00398 \ (0.00398107\ldots)$ ***A1***

[2 marks]

(c) Find how many times brighter Acubens is compared to Ceres. [2]

Markscheme

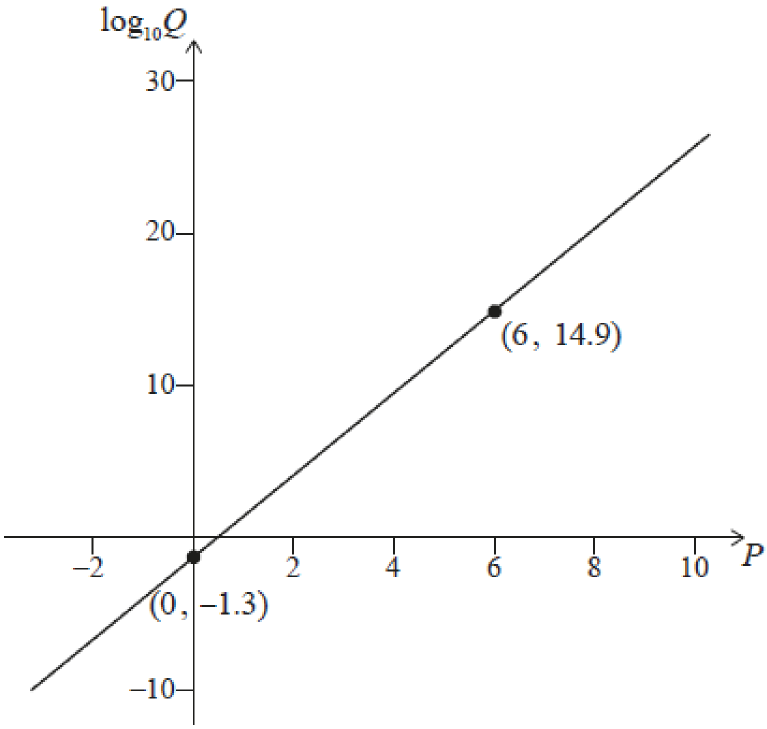
$\frac{0.0525}{0.00398107}$ ***(M1)***

$= 13.2 \ (13.1874\ldots)$ ***A1***

[2 marks]

12. [Maximum mark: 6] 22N.1.AHL.TZ0.13

Gen is investigating the relationship between two sets of data, labelled P and Q, that she collected. She created a scatter plot with P on the x-axis and $\log_{10} Q$ on the y-axis. Gen noticed that the points had a strong linear correlation, so she drew a line of best fit, as shown in the diagram. The line passes through the points $(0,\ -1.3)$ and $(6,\ 14.9)$.



(a) Find an equation for Q in terms of P. [3]

Markscheme

Gradient $= \frac{14.9+1.3}{6} (= 2.7)$ *(M1)*

$\log_{10} Q = 2.7P - 1.3$ *(A1)*

$Q = 10^{2.7P-1.3}$ **OR**
$Q = 0.0501 \times 10^{2.7P}$ $(= 0.0501187\ldots \times 10^{2.7P})$ *A1*

[3 marks]

Gen also investigates the relationship between the same data, Q, and some new data, R. She believes that the data can be modelled by $Q = a \ln R + b$ and she decides to create a scatter plot to verify her belief.

(b) State what expression Gen should plot on each axis to verify her belief. [1]

Markscheme

$\ln R$ on one axis and Q on the other axis ***A1***

[1 mark]

The scatter plot has a linear relationship and Gen finds $a = 4.3$ and $b = 12.1$.

(c) Find an equation for P in terms of R. [2]

Markscheme

$\log_{10}(4.3 \ln R + 12.1) = 2.7P - 1.3$ **OR**
$10^{2.7P-1.3} = 4.3 \ln R + 12.1$ ***(M1)***

$P = \frac{\log_{10}(4.3 \ln R+12.1)+1.3}{2.7}$ ***A1***

[2 marks]