Matrices revision 2 [77 marks]

1. [Maximum mark: 12] EXM.2.AHL.TZ0.15

The matrix A is defined by $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

(a) Describe fully the geometrical transformation represented by A. [5]

Markscheme

stretch ***A1***

scale factor 3, ***A1***

y-axis invariant (condone parallel to the x-axis) ***A1***

and

stretch, scale factor 2, ***A1***

x-axis invariant (condone parallel to the y-axis) ***A1***

[5 marks]

Pentagon, P, which has an area of 7 cm^2, is transformed by A.

(b) Find the area of the image of P. [2]

Markscheme

$\det(A) = 6$ ***A1***

$7 \times 6 = 42\,\text{cm}^2$ ***A1***

[2 marks]

The matrix B is defined by $B = \frac{1}{2}\begin{pmatrix} 3\sqrt{3} & 3 \\ -2 & 2\sqrt{3} \end{pmatrix}$.

B represents the combined effect of the transformation represented by a matrix X, followed by the transformation represented by A.

(c) Find the matrix X. [3]

Markscheme

$B = AX$ ***(A1)***

$X = A^{-1}B$ ***(M1)***

$$X = \begin{pmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{pmatrix} \left(= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \right)$$ ***A1***

[3 marks]

(d) Describe fully the geometrical transformation represented by X. [2]

Markscheme

Rotation ***A1***

clockwise by 30° about the origin ***A1***

[2 marks]

2. [Maximum mark: 16] EXM.2.AHL.TZ0.7

Let S_n be the sum of the first n terms of the arithmetic series 2 + 4 + 6 + ….

(a.i) Find S_4. [1]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$S_4 = 20$ ***A1 N1***

[1 mark]

(a.ii) Find S_{100}. [3]

Markscheme

$u_1 = 2, d = 2$ ***(A1)***

Attempting to use formula for S_n ***M1***

$S_{100} = 10100$ ***A1 N2***

[3 marks]

Let $\boldsymbol{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(b.i) Find $\boldsymbol{M}^2$. [2]

Markscheme

$\boldsymbol{M}^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ ***A2 N2***

[2 marks]

(b.ii) Show that $\boldsymbol{M}^3 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$. [3]

Markscheme

For writing $\boldsymbol{M}^3$ as $\boldsymbol{M}^2 \times \boldsymbol{M}$ or $\boldsymbol{M} \times \boldsymbol{M}^2$ $\left(\text{or} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\right)$ ***M1***

$\boldsymbol{M}^3 = \begin{pmatrix} 1+0 & 4+2 \\ 0+0 & 0+1 \end{pmatrix}$ ***A2***

$\boldsymbol{M}^3 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$ ***AG N0***

[3 marks]

It may now be assumed that $M^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$, for $n \geq 4$. The sum T_n is defined by

$T_n = M^1 + M^2 + M^3 + \ldots + M^n$.

(c.i) Write down M^4. [1]

Markscheme

$M^4 = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}$ ***A1 N1***

[1 mark]

(c.ii) Find T_4. [3]

Markscheme

$T_4 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}$ ***(M1)***

$= \begin{pmatrix} 4 & 20 \\ 0 & 4 \end{pmatrix}$ ***A1A1 N3***

[3 marks]

(d) Using your results from part (a) (ii), find T_{100}. [3]

Markscheme

$T_{100} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} + \ldots + \begin{pmatrix} 1 & 200 \\ 0 & 1 \end{pmatrix}$ ***(M1)***

$= \begin{pmatrix} 100 & 10100 \\ 0 & 100 \end{pmatrix}$ ***A1A1 N3***

[3 marks]

3. [Maximum mark: 16] EXM.2.AHL.TZ0.16

The matrices A and B are defined by $A = \begin{pmatrix} 3 & -2 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Describe fully the geometrical transformation represented by B. [2]

Markscheme

reflection in the y-axis ***A1A1***

[2 marks]

Triangle X is mapped onto triangle Y by the transformation represented by AB. The coordinates of triangle Y are (0, 0), (−30, −20) and (−16, 32).

(b) Find the coordinates of triangle X. [5]

Markscheme

$X = (AB)^{-1}Y$ ***M1***

EITHER

$AB = \begin{pmatrix} -3 & -2 \\ -2 & 4 \end{pmatrix}$, so $(AB)^{-1} = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{16} \end{pmatrix}$ ***M1A1***

OR

$X = B^{-1}A^{-1}Y$ ***M1A1***

THEN

$X = \begin{pmatrix} 0 & 10 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ ***(A1)***

So the coordinates are (0, 0), (10, 0) and (0, 8). ***A1***

[5 marks]

(c.i) Find the area of triangle X. [2]

Markscheme

$\frac{10\times 8}{2} = 40$ units^2 ***M1A1***

[2 marks]

(c.ii) Hence find the area of triangle Y. [3]

Markscheme

$\det\ (AB) = -16$ ***M1A1***

Area $= 40 \times 16 = 640$ units^2 ***A1***

[3 marks]

(d) Matrix A represents a combination of transformations:

A stretch, with scale factor 3 and y-axis invariant;
Followed by a stretch, with scale factor 4 and x-axis invariant;
Followed by a transformation represented by matrix C.

Find matrix C. [4]

Markscheme

A stretch, with scale factor 3 and y-axis invariant is given by $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ ***A1***

A stretch, with scale factor 4 and x-axis invariant is given by $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ ***A1***

So $C = A\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ \frac{2}{3} & 1 \end{pmatrix}$ ***M1A1***

[4 marks]

4. [Maximum mark: 11] EXM.2.AHL.TZ0.21

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

Given that $X = B - A^{-1}$ and $Y = B^{-1} - A$,

(a.i) find X and Y. [2]

Markscheme

$$X = B - A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \textbf{\textit{A1}}$$

$$Y = B^{-1} - A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad \textbf{\textit{A1}}$$

[2 marks]

(a.ii) does $X^{-1} + Y^{-1}$ have an inverse? Justify your conclusion. [3]

Markscheme

$$X^{-1} + Y^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \textbf{\textit{(A1)}}$$

$X^{-1} + Y^{-1}$ has no inverse ***A1***

as $\det(X^{-1} + Y^{-1}) = 0$ ***R1***

[3 marks]

You are told that $A^n = \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$, for $n \in \mathbb{Z}^+$.

Given that $(A^n)^{-1} = \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$, for $n \in \mathbb{Z}^+$,

(b.i) find x and y in terms of n. [5]

Markscheme

$$A^n(A^n)^{-1} = I \Rightarrow \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

M1

$$\Rightarrow \begin{pmatrix} 1 & x+n & y+nx+\frac{n(n+1)}{2} \\ 0 & 1 & x+n \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ***A1***

solve simultaneous equations to obtain

$x + n = 0$ and $y + nx + \frac{n(n+1)}{2} = 0$ ***M1***

$x = -n$ and $y = \frac{n(n-1)}{2}$ ***A1A1N2***

[5 marks]

(b.ii) and hence find an expression for $A^n + (A^n)^{-1}$. [1]

Markscheme

$$A^n + (A^n)^{-1} = \Rightarrow \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -n & \frac{n(n-1)}{2} \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & n^2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

A1

[1 mark]

5. [Maximum mark: 22] EXM.2.AHL.TZ0.23

Let $\boldsymbol{A} = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}$.

(a) Find the values of λ for which the matrix ($\boldsymbol{A} - \lambda\boldsymbol{I}$) is singular. [5]

Markscheme

$\boldsymbol{A} - \lambda\boldsymbol{I} = \begin{pmatrix} 3-\lambda & 1 \\ 4 & 3-\lambda \end{pmatrix}$ ***A1***

If $\boldsymbol{A} - \lambda\boldsymbol{I}$ is singular then det ($\boldsymbol{A} - \lambda\boldsymbol{I}$) = 0 ***(R1)***

det ($\boldsymbol{A} - \lambda\boldsymbol{I}$) $= (3-\lambda)^2 - 4\ (= \lambda^2 - 6\lambda + 5)$ ***(A1)***

Attempting to solve $(3-\lambda)^2 - 4 = 0$ or equivalent for λ ***M1***

λ = 1, 5 ***A1 N2***

Note: Candidates need both values of λ for the final ***A1***.

[5 marks]

Let $\boldsymbol{A}^2 + m\boldsymbol{A} + n\boldsymbol{I} = \boldsymbol{0}$ where $m, n \in \mathbb{Z}$ and $\boldsymbol{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(b.i) Find the value of m and of n. [5]

Markscheme

$\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}^2 + m\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix} + n\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ ***A1***

$\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}^2 = \begin{pmatrix} 13 & 6 \\ 24 & 13 \end{pmatrix}$ ***(A1)***

Forming any two independent equations ***M1***

(*eg* $6 + m = 0, 13 + 3m + n = 0$ or equivalent)

Note: Accept equations in matrix form.

Solving these two equations ***(M1)***

$m = -6$ and $n = 5$ ***A1 N2***

[5 marks]

(b.ii) Hence show that $\boldsymbol{I} = \frac{1}{5}\boldsymbol{A}(6\boldsymbol{I} - \boldsymbol{A})$. [4]

Markscheme

$\boldsymbol{A}^2 - 6\boldsymbol{A} + 5\boldsymbol{I} = \boldsymbol{0}$ ***(M1)***

$5\boldsymbol{I} = 6\boldsymbol{A} - \boldsymbol{A}^2$ ***A1***

$= \boldsymbol{A}(6\boldsymbol{I} - \boldsymbol{A})$ ***A1A1***

Note: Award ***A1*** for $\boldsymbol{A}$ and ***A1*** for $(6\boldsymbol{I} - \boldsymbol{A})$.

$\boldsymbol{I} = \frac{1}{5}\boldsymbol{A}(6\boldsymbol{I} - \boldsymbol{A})$ ***AG N0***

Special Case: Award ***M1A0A0A0*** only for candidates following alternative methods.

[5 marks]

(b.iii) Use the result from **part (b) (ii)** to explain why $\boldsymbol{A}$ is non-singular. [3]

Markscheme

METHOD 1

$\boldsymbol{I} = \frac{1}{5}\boldsymbol{A}(6\boldsymbol{I} - \boldsymbol{A}) = \boldsymbol{A} \times \frac{1}{5}(6\boldsymbol{I} - \boldsymbol{A})$ ***M1***

Hence by definition $\frac{1}{5}(6\boldsymbol{I} - \boldsymbol{A})$ is the inverse of $\boldsymbol{A}$. ***R1***

Hence $\boldsymbol{A}^{-1}$ exists and so $\boldsymbol{A}$ is non-singular ***R1 N0***

METHOD 2

As $\det \boldsymbol{I} = 1\ (\neq 0)$, then ***R1***

$\det \frac{1}{5}\boldsymbol{A}(6\boldsymbol{I} - \boldsymbol{A}) = \frac{1}{5} \det \boldsymbol{A} \times \det(6\boldsymbol{I} - \boldsymbol{A})\ (\neq 0)$ ***M1***

$\Rightarrow \det \boldsymbol{A} \neq 0$ and so $\boldsymbol{A}$ is non-singular. ***R1 N0***

[3 marks]

(c) Use the values from **part (b) (i)** to express $\boldsymbol{A}^4$ in the form $p\boldsymbol{A} + q\boldsymbol{I}$ where p, $q \in \mathbb{Z}$. [5]

Markscheme

METHOD 1

$\boldsymbol{A}^2 = 6\boldsymbol{A} - 5\boldsymbol{I}$ ***(A1)***

$\boldsymbol{A}^4 = (6\boldsymbol{A} - 5\boldsymbol{I})^2$ ***M1***

$= 36\boldsymbol{A}^2 - 60\boldsymbol{AI} + 25\boldsymbol{I}^2$ ***A1***

$= 36(6\boldsymbol{A} - 5\boldsymbol{I}) - 60\boldsymbol{A} + 25\boldsymbol{I}$ ***M1***

$= 156\boldsymbol{A} - 155\boldsymbol{I}$ $(p = 156, q = -155)$ ***A1 N0***

METHOD 2

$\boldsymbol{A}^2 = 6\boldsymbol{A} - 5\boldsymbol{I}$ ***(A1)***

$\boldsymbol{A}^3 = 6\boldsymbol{A}^2 - 5\boldsymbol{A}$ where $\boldsymbol{A}^2 = 6\boldsymbol{A} - 5\boldsymbol{I}$ ***M1***

$= 31\boldsymbol{A} - 30\boldsymbol{I}$ ***A1***

$\boldsymbol{A}^4 = 31\boldsymbol{A}^2 - 30\boldsymbol{A}$ where $\boldsymbol{A}^2 = 6\boldsymbol{A} - 5\boldsymbol{I}$ ***M1***

$= 156\boldsymbol{A} - 155\boldsymbol{I}$ $(p = 156, q = -155)$ ***A1 N0***

Note: Do not accept methods that evaluate A4 directly from $\boldsymbol{A}$.

[5 marks]