Matrices revision 2 [77 marks]

1. [Maximum mark: 12] EXM.2.AHL.TZ0.15

The matrix A is defined by $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

(a) Describe fully the geometrical transformation represented by A. [5]

Pentagon, P, which has an area of 7 cm^2, is transformed by A.

(b) Find the area of the image of P. [2]

The matrix B is defined by $B = \frac{1}{2}\begin{pmatrix} 3\sqrt{3} & 3 \\ -2 & 2\sqrt{3} \end{pmatrix}$.

B represents the combined effect of the transformation represented by a matrix X, followed by the transformation represented by A.

(c) Find the matrix X. [3]

(d) Describe fully the geometrical transformation represented by X. [2]

2. [Maximum mark: 16] EXM.2.AHL.TZ0.7

Let S_n be the sum of the first n terms of the arithmetic series 2 + 4 + 6 + ….

(a.i) Find S_4. [1]

(a.ii) Find S_{100}. [3]

Let $\boldsymbol{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(b.i) Find $\boldsymbol{M}^2$. [2]

(b.ii) Show that $\boldsymbol{M}^3 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$. [3]

It may now be assumed that $\boldsymbol{M}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$, for $n \geq 4$. The sum $\boldsymbol{T}_n$ is defined by

$\boldsymbol{T}_n = \boldsymbol{M}^1 + \boldsymbol{M}^2 + \boldsymbol{M}^3 + \ldots + \boldsymbol{M}^n$.

(c.i) Write down $\boldsymbol{M}^4$. [1]

(c.ii) Find $\boldsymbol{T}_4$. [3]

(d) Using your results from part (a) (ii), find $\boldsymbol{T}_{100}$. [3]

3. [Maximum mark: 16] EXM.2.AHL.TZ0.16

The matrices A and B are defined by $A = \begin{pmatrix} 3 & -2 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Describe fully the geometrical transformation represented by B. [2]

Triangle X is mapped onto triangle Y by the transformation represented by AB.
The coordinates of triangle Y are (0, 0), (−30, −20) and (−16, 32).

(b) Find the coordinates of triangle X. [5]

(c.i) Find the area of triangle X. [2]

(c.ii) Hence find the area of triangle Y. [3]

(d) Matrix A represents a combination of transformations:

A stretch, with scale factor 3 and y-axis invariant;
Followed by a stretch, with scale factor 4 and x-axis invariant;

Followed by a transformation represented by matrix C.

[4]

Find matrix C.

4. [Maximum mark: 11] EXM.2.AHL.TZ0.21

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

Given that $\boldsymbol{X} = \boldsymbol{B} - \boldsymbol{A}^{-1}$ and $\boldsymbol{Y} = \boldsymbol{B}^{-1} - \boldsymbol{A}$,

(a.i) find $\boldsymbol{X}$ and $\boldsymbol{Y}$. [2]

(a.ii) does $\boldsymbol{X}^{-1} + \boldsymbol{Y}^{-1}$ have an inverse? Justify your conclusion. [3]

You are told that $A^n = \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$, for $n \in \mathbb{Z}^+$.

Given that $(A^n)^{-1} = \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$, for $n \in \mathbb{Z}^+$,

(b.i) find x and y in terms of n. [5]

(b.ii) and hence find an expression for $A^n + (A^n)^{-1}$. [1]

5. [Maximum mark: 22] EXM.2.AHL.TZ0.23

Let $\boldsymbol{A} = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}$.

(a) Find the values of λ for which the matrix $(\boldsymbol{A} - \lambda\boldsymbol{I})$ is singular. [5]

Let $\boldsymbol{A}^2 + m\boldsymbol{A} + n\boldsymbol{I} = \boldsymbol{0}$ where $m, n \in \mathbb{Z}$ and $\boldsymbol{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(b.i) Find the value of m and of n. [5]

(b.ii) Hence show that $\boldsymbol{I} = \frac{1}{5}\boldsymbol{A}(6\boldsymbol{I} - \boldsymbol{A})$. [4]

(b.iii) Use the result from **part (b) (ii)** to explain why $\boldsymbol{A}$ is non-singular. [3]

(c) Use the values from **part (b) (i)** to express $\boldsymbol{A}^4$ in the form $p\boldsymbol{A} + q\boldsymbol{I}$ where $p, q \in \mathbb{Z}$. [5]