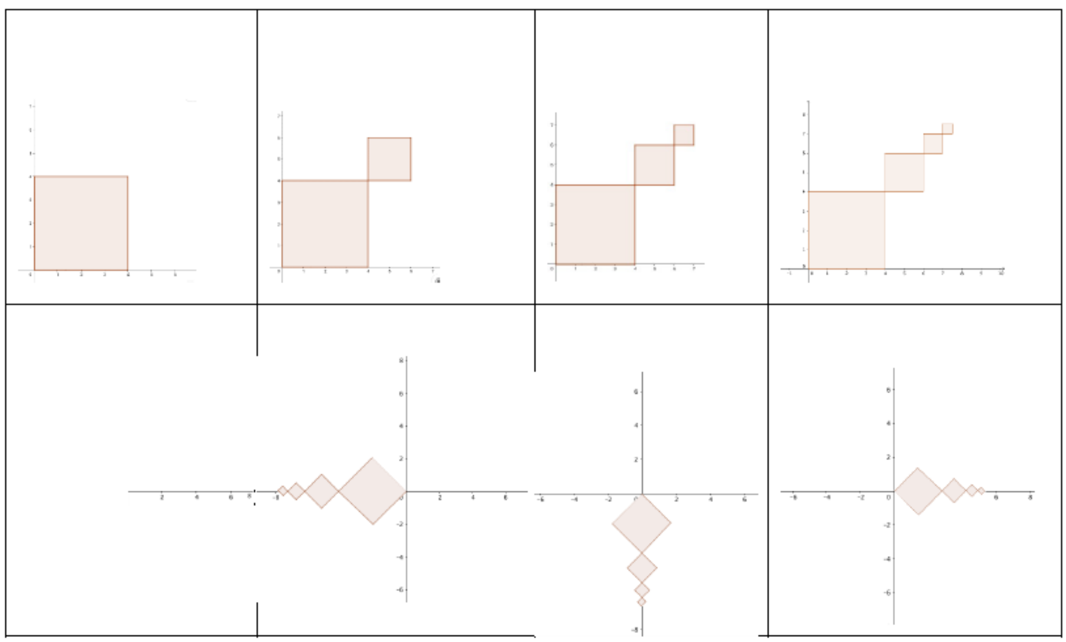
Matrices revision 3 [82 marks]

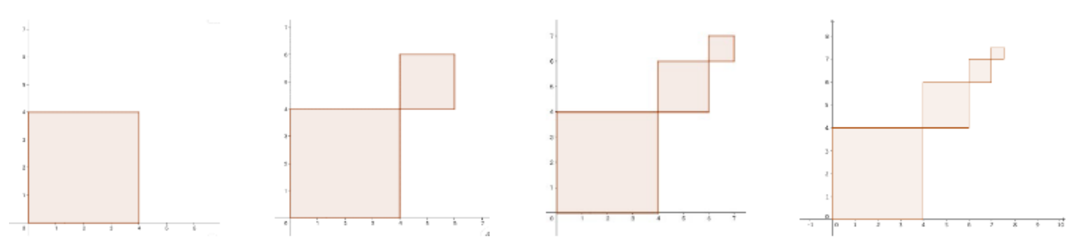
1. [Maximum mark: 29] EXN.3.AHL.TZ0.2

A graphic designer, Ben, wants to create an animation in which a sequence of squares is created by a composition of successive enlargements and translations and then rotated about the origin and reduced in size.

Ben outlines his plan with the following storyboards.



The first four frames of the animation are shown below in greater detail.



The sides of each successive square are one half the size of the adjacent larger square. Let the sequence of squares be $U_0,\ U_1,\ U_2,\ \ldots$

The first square, U_0, has sides of length $4\,\text{cm}$.

(a) Find an expression for the width of U_n in centimetres. [2]

Ben decides the animation will continue as long as the width of the square is greater than the width of one pixel.

(b) Given the width of a pixel is approximately $0.025\,\text{cm}$, find the number of squares in the final image. [3]

Ben decides to generate the squares using the transformation

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \boldsymbol{A}_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \boldsymbol{b}_n$$

where $\boldsymbol{A}_n$ is a 2×2 matrix that represents an enlargement, $\boldsymbol{b}_n$ is a 2×1 column vector that represents a translation, $(x_0,\, y_0)$ is a point in $\boldsymbol{U}_0$ and $(x_n,\, y_n)$ is its image in $\boldsymbol{U}_n$.

(c.i) Write down $\boldsymbol{A}_1$. [1]

(c.ii) Write down $\boldsymbol{A}_n$, in terms of n. [1]

By considering the case where $(x_0,\, y_0)$ is $(0,\, 0)$,

(d.i) state the coordinates, $(x_1,\, y_1)$, of its image in U_1. [1]

(d.ii) hence find $\boldsymbol{b}_1$. [2]

(d.iii) show that $\boldsymbol{b}_n = \begin{pmatrix} 8(1 - 2^{-n}) \\ 8(1 - 2^{-n}) \end{pmatrix}$. [3]

(e) Hence or otherwise, find the coordinates of the top left-hand corner in U_7. [3]

Once the image of squares has been produced, Ben wants to continue the animation by rotating the image counter clockwise about the origin and having it reduce in size during the rotation.

Let E_θ be the enlargement matrix used when the original sequence of squares has been rotated through θ degrees.

Ben decides the enlargement scale factor, s, should be a linear function of the angle, θ, and after a rotation of 360° the sequence of squares should be half of its original length.

(f.i) Find, s, in the form $s(\theta) = m\theta + c$. [4]

(f.ii) Write down E_θ. [1]

(f.iii) Hence find the image of $(1,\ 1)$ after it is rotated 135° and enlarged. [4]

(g) Find the value of θ at which the enlargement scale factor equals zero. [1]

(h) After the enlargement scale factor equals zero, Ben continues to rotate the image for another two revolutions.

Describe the animation for these two revolutions, stating the final position of the sequence of squares. [3]

2. [Maximum mark: 7] EXM.1.AHL.TZ0.17

Sue sometimes goes out for lunch. If she goes out for lunch on a particular day then the probability that she will go out for lunch on the following day is 0.4. If she does not go out for lunch on a particular day then the probability she will go out for lunch on the following day is 0.3.

(a) Write down the transition matrix for this Markov chain. [2]

(b) We know that she went out for lunch on a particular Sunday, find the probability that she went out for lunch on the following Tuesday. [2]

(c) Find the steady state probability vector for this Markov chain. [3]

3. [Maximum mark: 8] 24M.1.AHL.TZ1.11

Let $\boldsymbol{R}(\alpha)$ be the matrix representing a rotation, counter-clockwise (anticlockwise) about the origin, through an angle of α.

(a) Write down $\boldsymbol{R}(2\alpha)$ as a 2×2 matrix. [2]

(b) Calculate $\boldsymbol{R}(\alpha) \times \boldsymbol{R}(\alpha)$. [2]

(c) Use your answers from part (a) and part (b) to

(c.i) explain why $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$. [2]

(c.ii) show that $\cos(2\alpha) = 1 - 2\sin^2(\alpha)$. [2]

4. [Maximum mark: 9] 24M.1.AHL.TZ1.17

Phoebe opens a coffee shop, near to a well-established Apollo coffee shop.
After being open for a few months, Phoebe notices that

- 10% of customers who preferred the Apollo coffee shop in one month preferred her coffee shop the following month.
- 25% of customers who preferred her coffee shop in one month preferred the Apollo coffee shop the following month.

She decides to show these changes in the following transition matrix.

$$\begin{pmatrix} 0.9 & 0.25 \\ 0.1 & 0.75 \end{pmatrix}$$

The two eigenvalues for this matrix are 1 and 0.65. An eigenvector corresponding to the eigenvalue of 1 is $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

(a) Find an eigenvector corresponding to the eigenvalue of 0.65. [2]

A diagonal matrix of eigenvalues is $\boldsymbol{D} = \begin{pmatrix} 0.65 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) Write down an expression for $\boldsymbol{D}^n$, giving your answer as a 2×2 matrix in terms of n. [1]

When Phoebe's coffee shop first opened, the Apollo shop had 7000 customers the previous month.

(c) Assuming all 7000 customers continue to go to one of these coffee shops, find an expression for the number that will favour Phoebe's coffee shop after n months. [6]

5. [Maximum mark: 14] 23N.2.AHL.TZ0.6

François is a video game designer. He designs his games to take place in two dimensions, relative to an origin O. In one of his games, an object travels on a straight line L_1 with vector equation

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(a) Write down L_1 in the form $x = x_0 + \lambda l$ and $y = y_0 + \lambda m$, where $l,\ m \in \mathbb{Z}$. [1]

François uses the matrix $\boldsymbol{T} = \begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix}$ to transform L_1 into a new straight line L_2. The object will then travel along L_2.

(b) Find the vector equation of L_2. [4]

François knows that the transformation given by matrix $\boldsymbol{T}$ is made up of the following three separate transformations (in the order listed):

- A rotation of $\frac{\pi}{4}$, anticlockwise (counter-clockwise) about the origin O
- An enlargement of scale factor $5\sqrt{2}$, centred at O
- A reflection in the straight line $y = mx$, where $m = \tan\alpha$, $0 \leq \alpha < \pi$

(c) Write down the matrix that represents

(c.i) the rotation. [1]

(c.ii) the enlargement. [1]

(d) The matrix $\boldsymbol{R}$ represents the reflection. Write down $\boldsymbol{R}$ in terms of α. [1]

(e) Given that $\boldsymbol{T} = \boldsymbol{R}\boldsymbol{X}$,

(e.i) use your answers to part (c) to find matrix $\boldsymbol{X}$. [2]

(e.ii) hence, find the value of α. [4]

6. [Maximum mark: 7] 23M.1.AHL.TZ2.13

The matrices $\boldsymbol{P} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{Q} = \begin{pmatrix} -4 & 1 \\ 1 & 3 \end{pmatrix}$ represent two transformations.

A triangle T is transformed by $\boldsymbol{P}$, and this image is then transformed by $\boldsymbol{Q}$ to form a new triangle, T'.

(a) Find the single matrix that represents the transformation $T' \rightarrow T$, which will undo the transformation described above. [4]

The area of T' is $273\ \mathrm{cm}^2$.

(b) Using your answer to part (a), or otherwise, determine the area of T. [3]

7. [Maximum mark: 8] 22N.1.AHL.TZ0.9

The transformation T is represented by the matrix $\boldsymbol{M} = \begin{pmatrix} 2 & -4 \\ 3 & 1 \end{pmatrix}$.

A pentagon with an area of $12\,\text{cm}^2$ is transformed by T.

(a) Find the area of the image of the pentagon. [2]

Under the transformation T, the image of point X has coordinates $(2t - 3,\ 6 - 5t)$, where $t \in \mathbb{R}$.

(b) Find, in terms of t, the coordinates of X. [6]