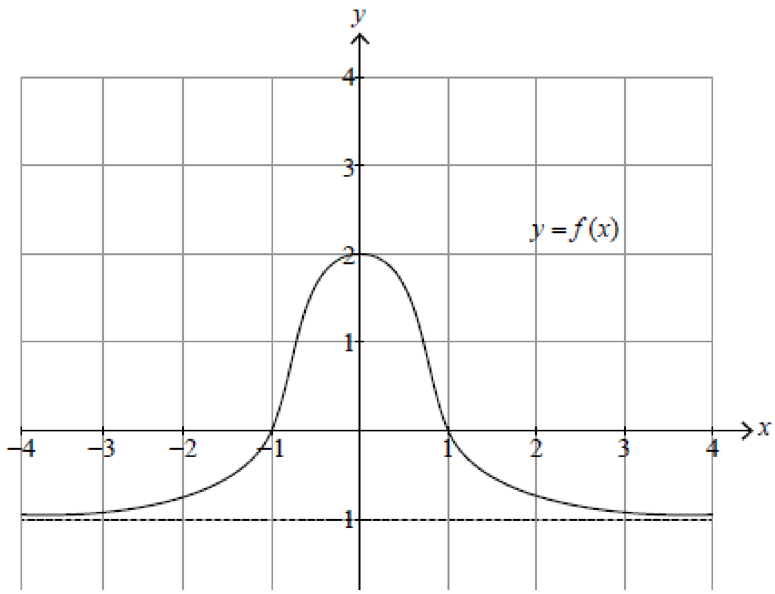
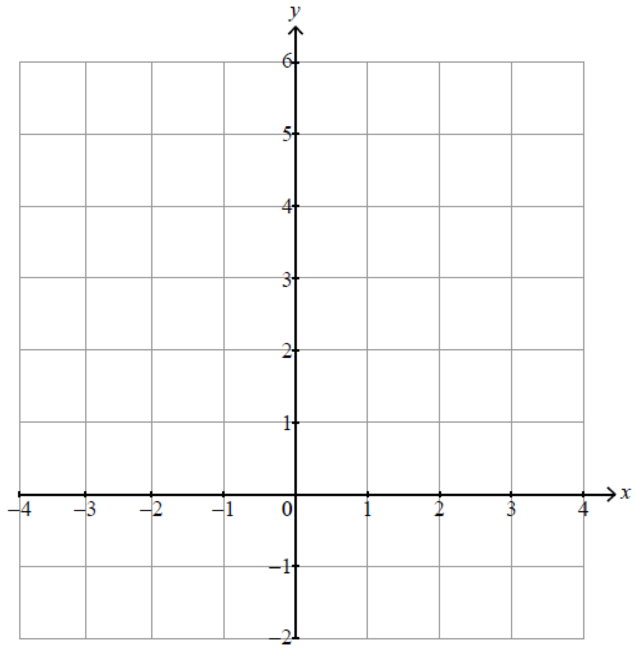
Revision set (Paper 1) [297 marks]

1. [Maximum mark: 5] SPM.1.AHL.TZ0.4

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the x-axis at $x = -1$ and $x = 1$, and the y-axis at $y = 2$.

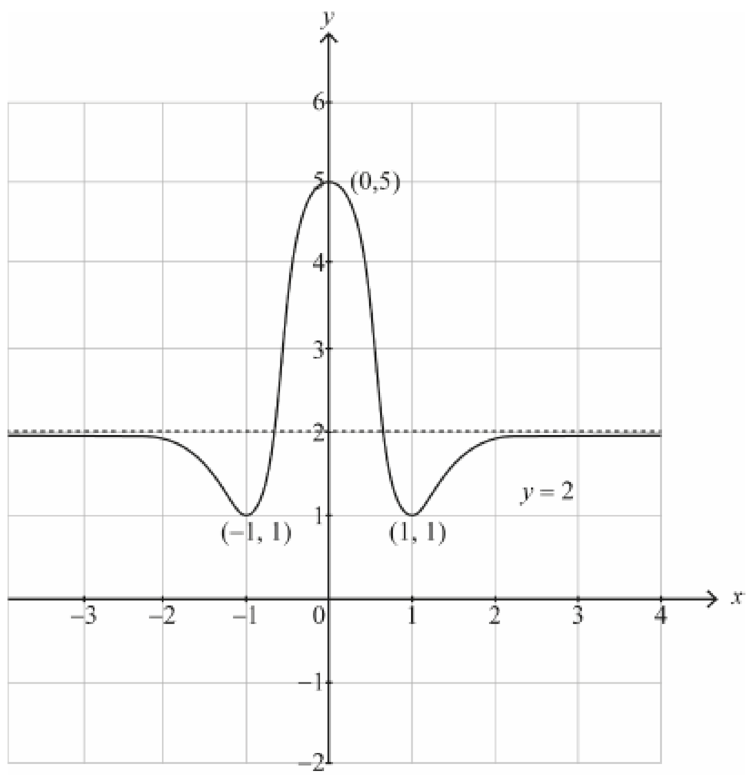


On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.



[5]

Markscheme



no y values below 1 ***A1***

horizontal asymptote at $y = 2$ with curve approaching from below as $x \to \pm\infty$ ***A1***

(±1,1) local minima ***A1***

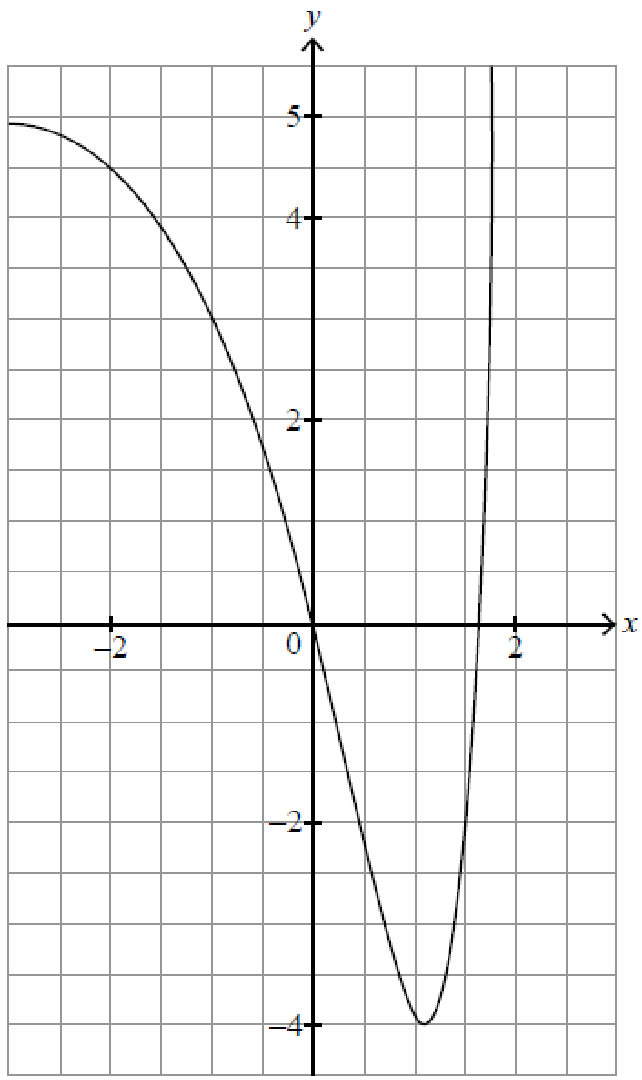
(0,5) local maximum ***A1***

smooth curve and smooth stationary points ***A1***

[5 marks]

2. [Maximum mark: 8] SPM.1.AHL.TZ0.9

The function f is defined by $f(x) = \mathrm{e}^{2x} - 6\mathrm{e}^{x} + 5,\ x \in \mathbb{R},\ x \leqslant a$. The graph of $y = f(x)$ is shown in the following diagram.



(a) Find the largest value of a such that f has an inverse function. [3]

Markscheme

attempt to differentiate and set equal to zero ***M1***

$f'(x) = 2e^{2x} - 6e^{x} = 2e^{x}(e^{x} - 3) = 0$ ***A1***

minimum at $x = \ln 3$

$a = \ln 3$ ***A1***

[3 marks]

(b) For this value of a, find an expression for $f^{-1}(x)$, stating its domain. [5]

Markscheme

Note: Interchanging x and y can be done at any stage.

$y = (e^x - 3)^2 - 4$ ***(M1)***

$e^x - 3 = \pm\sqrt{y+4}$ ***A1***

as $x \leqslant \ln 3$, $x = \ln\left(3 - \sqrt{y+4}\right)$ ***R1***

so $f^{-1}(x) = \ln\left(3 - \sqrt{x+4}\right)$ ***A1***

domain of f^{-1} is $x \in \mathbb{R}, -4 \leqslant x < 5$ ***A1***

[5 marks]

3. [Maximum mark: 5] EXN.1.AHL.TZ0.6

Use l'Hôpital's rule to determine the value of $\lim_{x\to 0}\left(\frac{2x\cos(x^2)}{5\tan x}\right)$. [5]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

attempts to apply l'Hôpital's rule on $\lim_{x\to 0}\left(\frac{2x\cos(x^2)}{5\tan x}\right)$ **M1**

$= \lim_{x\to 0}\left(\frac{2\cos(x^2) - 4x^2\sin(x^2)}{5\sec^2 x}\right)$ **M1A1A1**

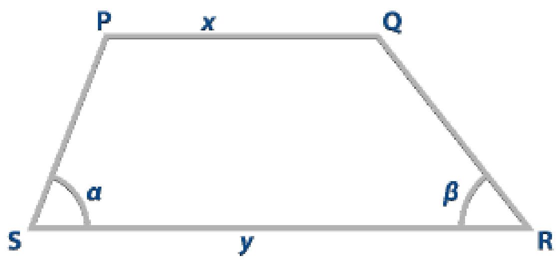
Note: Award **M1** for attempting to use product and chain rule differentiation on the numerator, **A1** for a correct numerator and **A1** for a correct denominator. The awarding of **A1** for the denominator is independent of the **M1**.

$= \frac{2}{5}$ **A1**

[5 marks]

4. [Maximum mark: 5] EXN.1.AHL.TZ0.7

Consider quadrilateral PQRS where $[\mathrm{PQ}]$ is parallel to $[\mathrm{SR}]$.



In PQRS, $\mathrm{PQ} = x$, $\mathrm{SR} = y$, $\mathrm{R\hat{S}P} = \alpha$ and $\mathrm{Q\hat{R}S} = \beta$.

Find an expression for PS in terms of $x,\ y,\ \sin\beta$ and $\sin(\alpha+\beta)$. [5]

Markscheme

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METHOD 1

from vertex P, draws a line parallel to $[\mathrm{QR}]$ that meets $[\mathrm{SR}]$ at a point X **(M1)**

uses the sine rule in $\Delta\mathrm{PSX}$ **M1**

$\frac{\mathrm{PS}}{\sin\beta} = \frac{y-x}{\sin(180^\circ-\alpha-\beta)}$ **A1**

$\sin(180^\circ-\alpha-\beta) = \sin(\alpha+\beta)$ **(A1)**

$\mathrm{PS} = \frac{(y-x)\sin\beta}{\sin(\alpha+\beta)}$ **A1**

METHOD 2

let the height of quadrilateral PQRS be h

$h = \mathrm{PS}\ \sin\alpha$ **A1**

attempts to find a second expression for h **M1**

$h=(y-x-\mathrm{PS}\ \cos\alpha)\tan\beta$

$\mathrm{PS}\ \sin\alpha=(y-x-\mathrm{PS}\ \cos\alpha)\tan\beta$

writes $\tan\beta$ as $\frac{\sin\beta}{\cos\beta}$, multiplies through by $\cos\beta$ and expands the RHS **M1**

$\mathrm{PS}\ \sin\alpha\cos\beta=(y-x)\sin\beta-\mathrm{PS}\ \cos\alpha\sin\beta$

$\mathrm{PS}=\frac{(y-x)\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$ **A1**

$\mathrm{PS}=\frac{(y-x)\sin\beta}{\sin(\alpha+\beta)}$ **A1**

[5 marks]

5. [Maximum mark: 21] EXN.1.AHL.TZ0.11

A function f is defined by $f(x)=\frac{3}{x^2+2},\ x\in\mathbb{R}$.

(a) Sketch the curve $y=f(x)$, clearly indicating any asymptotes with their equations and stating the coordinates of any points of intersection with the axes. [4]

Markscheme

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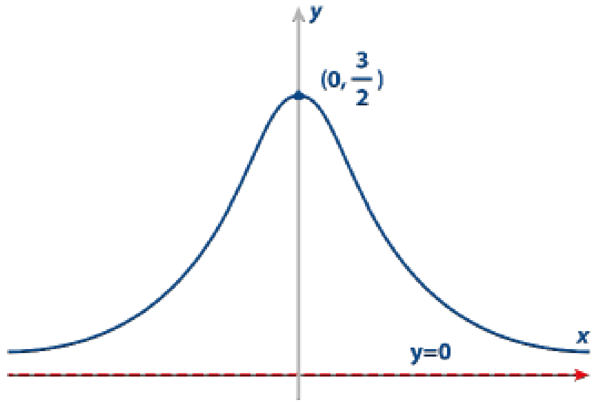


a curve symmetrical about the y-axis with correct concavity that has a local maximum point on the positive y-axis **A1**

a curve clearly showing that $y \to 0$ as $x \to \pm\infty$ **A1**

$\left(0, \frac{3}{2}\right)$ **A1**

horizontal asymptote $y = 0$ (x-axis) **A1**

[4 marks]

The region R is bounded by the curve $y = f(x)$, the x-axis and the lines $x = 0$ and $x = \sqrt{6}$. Let A be the area of R.

(b) Show that $A = \frac{\sqrt{2}\pi}{2}$. [4]

Markscheme

attempts to find $\int \frac{3}{x^2+2} \mathrm{d}x$ **(M1)**

$= \left[\frac{3}{\sqrt{2}} \arctan \frac{x}{\sqrt{2}}\right]$ **A1**

Note: Award **M1A0** for obtaining $\left[k \ \arctan \frac{x}{\sqrt{2}}\right]$ where $k \neq \frac{3}{\sqrt{2}}$.

Note: Condone the absence of or use of incorrect limits to this stage.

$= \frac{3}{\sqrt{2}}\left(\arctan \sqrt{3} - \arctan 0\right)$ **(M1)**

$= \frac{3}{\sqrt{2}} \times \frac{\pi}{3}\left(= \frac{\pi}{\sqrt{2}}\right)$ **A1**

$A = \frac{\sqrt{2}\pi}{2}$ **AG**

[4 marks]

The line $x = k$ divides R into two regions of equal area.

(c) Find the value of k. [4]

Markscheme

METHOD 1

EITHER

$\int\limits_0^k \frac{3}{x^2+2}\mathrm{d}x = \frac{\sqrt{2}\pi}{4}$

$\frac{3}{\sqrt{2}}\arctan \frac{k}{\sqrt{2}} = \frac{\sqrt{2}\pi}{4}$ **(M1)**

OR

$\int\limits_k^{\sqrt{6}} \frac{3}{x^2+2}\mathrm{d}x = \frac{\sqrt{2}\pi}{4}$

$\frac{3}{\sqrt{2}}\left(\arctan \sqrt{3} - \arctan \frac{k}{\sqrt{2}}\right) = \frac{\sqrt{2}\pi}{4}$ **(M1)**

$\arctan \sqrt{3} - \arctan \frac{k}{\sqrt{2}} = \frac{\pi}{6}$

THEN

$\arctan \frac{k}{\sqrt{2}} = \frac{\pi}{6}$ **A1**

$\frac{k}{\sqrt{2}} = \tan \frac{\pi}{6} \left(= \frac{1}{\sqrt{3}}\right)$ **A1**

$k = \frac{\sqrt{6}}{3} \left(= \sqrt{\frac{2}{3}}\right)$ **A1**

METHOD 2

$\int_0^k \frac{3}{x^2+2} \mathrm{d}x = \int_k^{\sqrt{6}} \frac{3}{x^2+2} \mathrm{d}x$

$\frac{3}{\sqrt{2}} \arctan \frac{k}{\sqrt{2}} = \frac{3}{\sqrt{2}} \left(\arctan \sqrt{3} - \arctan \frac{k}{\sqrt{2}}\right)$ **(M1)**

$\arctan \frac{k}{\sqrt{2}} = \frac{\pi}{6}$ **A1**

$\frac{k}{\sqrt{2}} = \tan \frac{\pi}{6} \left(= \frac{1}{\sqrt{3}}\right)$ **A1**

$k = \frac{\sqrt{6}}{3} \left(= \sqrt{\frac{2}{3}}\right)$ **A1**

[4 marks]

Let m be the gradient of a tangent to the curve $y = f(x)$.

(d) Show that $m = -\frac{6x}{(x^2+2)^2}$. [2]

Markscheme

attempts to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3}{x^2+2}\right)$ **(M1)**

$= (3)(-1)(2x)\left(x^2+2\right)^{-2}$ **A1**

so $m = -\frac{6x}{(x^2+2)^2}$ **AG**

[2 marks]

(e) Show that the maximum value of m is $\frac{27}{32}\sqrt{\frac{2}{3}}$. [7]

Markscheme

attempts product rule or quotient rule differentiation **M1**

EITHER

$\frac{\mathrm{d}m}{\mathrm{d}x} = (-6x)(-2)(2x)\left(x^2+2\right)^{-3} + \left(x^2+2\right)^{-2}(-6)$ **A1**

OR

$\frac{\mathrm{d}m}{\mathrm{d}x} = \frac{\left(x^2+2\right)^2(-6)-(-6x)(2)(2x)\left(x^2+2\right)}{\left(x^2+2\right)^4}$ **A1**

Note: Award **A0** if the denominator is incorrect. Subsequent marks can be awarded.

THEN

attempts to express their $\frac{\mathrm{d}m}{\mathrm{d}x}$ as a rational fraction with a factorized numerator **M1**

$\frac{\mathrm{d}m}{\mathrm{d}x} = \frac{6\left(x^2+2\right)\left(3x^2-2\right)}{\left(x^2+2\right)^4}\left(= \frac{6\left(3x^2-2\right)}{\left(x^2+2\right)^3}\right)$

attempts to solve their $\frac{\mathrm{d}m}{\mathrm{d}x} = 0$ for x **M1**

$x = \pm\sqrt{\frac{2}{3}}$ **A1**

from the curve, the maximum value of m occurs at $x = -\sqrt{\frac{2}{3}}$ **R1**

(the minimum value of m occurs at $x = \sqrt{\frac{2}{3}}$)

Note: Award **R1** for any equivalent valid reasoning.

maximum value of m is $-\frac{6\left(-\sqrt{\frac{2}{3}}\right)}{\left(\left(-\sqrt{\frac{2}{3}}\right)^2+2\right)^2}$ **A1**

leading to a maximum value of $\frac{27}{32}\sqrt{\frac{2}{3}}$ **AG**

[7 marks]

6. [Maximum mark: 8] EXM.1.AHL.TZ0.1

Let $f(x) = \frac{1}{1-x^2}$ for $-1 < x < 1$. Use partial fractions to find $\int f(x)\,dx$. [8]

Markscheme

$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} \equiv \frac{A}{1-x} + \frac{B}{1+x}$ ***M1M1A1***

$\Rightarrow 1 \equiv A(1+x) + B(1-x) \Rightarrow A = B = \frac{1}{2}$ ***M1A1A1***

$\int \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x} dx = \frac{-1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) + c \quad \left(= \ln k\sqrt{\frac{1+x}{1-x}}\right)$ ***M1A1***

[8 marks]

7. [Maximum mark: 9] EXM.1.AHL.TZ0.6

Let $f(x) = \frac{x^2-10x+5}{x+1},\ x \in \mathbb{R},\ x \neq -1.$

(a) Find the co-ordinates of all stationary points. [4]

Markscheme

$f'(x) = \frac{(2x-10)(x+1)-(x^2-10x+5)1}{(x+1)^2}$ ***M1***

$f'(x) = 0 \Rightarrow x^2 + 2x - 15 = 0 \Rightarrow (x+5)(x-3) = 0$ ***M1***

Stationary points are $(-5,\ -20)$ and $(3,\ -4)$ ***A1A1***

[4 marks]

(b) Write down the equation of the vertical asymptote. [1]

Markscheme

$x = -1$ ***A1***

[1 mark]

(c) With justification, state if each stationary point is a minimum, maximum or horizontal point of inflection. [4]

Markscheme

Looking at the nature table

x		-5		-1		3	
$f'(x)$	+ve	0	-ve	undefined	-ve	0	+ve

M1A1

$(-5,\ -20)$ is a max and $(3,\ -4)$ is a min ***A1A1***

[4 marks]

8. [Maximum mark: 11] EXM.1.AHL.TZ0.2

Consider the integral $\int\limits_1^t \frac{-1}{x+x^2}\ dx$ for $t > 1$.

(a) Very briefly, explain why the value of this integral must be negative. [1]

Markscheme

The numerator is negative but the denominator is positive. Thus the integrand is negative and so the value of the integral will be negative. ***R1AG***

[1 mark]

(b) Express the function $f(x) = \frac{-1}{x+x^2}$ in partial fractions. [6]

Markscheme

$\frac{-1}{x+x^2} = \frac{-1}{(1+x)x} \equiv \frac{A}{1+x} + \frac{B}{x}$ ***M1M1A1***

$\Rightarrow -1 \equiv Ax + B(1+x) \Rightarrow A = 1,\ B = -1$ ***M1A1***

$\frac{-1}{x+x^2} \equiv \frac{1}{1+x} + \frac{-1}{x}$ ***A1***

[6 marks]

(c) Use parts (a) and (b) to show that $\ln(1+t) - \ln t < \ln 2$. [4]

Markscheme

$\int_1^t \frac{1}{1+x} + \frac{-1}{x} dx = [\ln(1+x) - \ln x]_1^t = \ln(1+t) - \ln t - \ln 2$ ***M1A1A1***

Hence $\ln(1+t) - \ln t - \ln 2 < 0 \Rightarrow \ln(1+t) - \ln t < \ln 2$ ***R1AG***

[4 marks]

9. [Maximum mark: 8] EXM.1.AHL.TZ0.4

Let $f(x) = \frac{2x+6}{x^2+6x+10},\ x \in \mathbb{R}$.

(a) Show that $f(x)$ has no vertical asymptotes. [3]

Markscheme

$x^2 + 6x + 10 = x^2 + 6x + 9 + 1 = (x+3)^2 + 1$ ***M1A1***

So the denominator is never zero and thus there are no vertical asymptotes. (or use of discriminant is negative) ***R1***

[3 marks]

(b) Find the equation of the horizontal asymptote. [2]

Markscheme

$x \to \pm\infty,\ f(x) \to 0$ so the equation of the horizontal asymptote is $y = 0$ ***M1A1***

[2 marks]

(c) Find the exact value of $\int\limits_0^1 f(x)\,dx$, giving the answer in the form $\ln q,\ q \in \mathbb{Q}$. [3]

Markscheme

$\int\limits_0^1 \frac{2x+6}{x^2+6x+10}\,dx = \left[\ln\left(x^2+6x+10\right)\right]_0^1 = \ln 17 - \ln 10 = \ln \frac{17}{10}$ ***M1A1A1***

[3 marks]

10. [Maximum mark: 9] EXM.1.AHL.TZ0.5

Let $f(x) = \frac{2x^2-5x-12}{x+2},\ x \in \mathbb{R},\ x \neq -2.$

(a) Find all the intercepts of the graph of $f(x)$ with both the x and y axes. [4]

Markscheme

$x = 0 \Rightarrow y = -6$ intercept on the y axes is (0, −6) ***A1***

$2x^2 - 5x - 12 = 0 \Rightarrow (2x+3)(x-4) = 0 \Rightarrow x = \frac{-3}{2} \text{ or } 4$ ***M1***

intercepts on the x axes are $\left(\frac{-3}{2},\ 0\right) \text{ and } (4,\ 0)$ ***A1A1***

[4 marks]

(b) Write down the equation of the vertical asymptote. [1]

Markscheme

$x = -2$ ***A1***

[1 mark]

(c) As $x \to \pm\infty$ the graph of $f(x)$ approaches an oblique straight line asymptote.

Divide $2x^2 - 5x - 12$ by $x + 2$ to find the equation of this asymptote. [4]

Markscheme

$f(x) = 2x - 9 + \frac{6}{x+2}$ ***M1A1***

So equation of asymptote is $y = 2x - 9$ ***M1A1***

[4 marks]

11. [Maximum mark: 5] 24M.1.AHL.TZ1.2

It is given that $\log_{10} a = \frac{1}{3}$, where $a > 0$.

Find the value of

(a) $\log_{10}\left(\frac{1}{a}\right)$; [2]

Markscheme

$\log_{10} 1 - \log_{10} a$ OR $\log_{10} a^{-1} = -\log_{10} a$ OR $\log_{10} 10^{-\frac{1}{3}}$ OR $10^x = \frac{1}{10^{\frac{1}{3}}}$ ***(A1)***

$= -\frac{1}{3}$ ***A1***

[2 marks]

(b) $\log_{1000} a$. [3]

Markscheme

$\frac{\log_{10} a}{\log_{10} 1000}$ OR $\frac{1}{3}\log_{1000} 10$ OR $\log_{1000} \sqrt[3]{1000^{\frac{1}{3}}}$ OR $10^{\frac{1}{3}} = 1000^x \left(= \left(10^3\right)^x\right)$ ***(A1)***

$\frac{\log_{10} a}{3}$ OR $\frac{1}{3} \log_{1000} 1000^{\frac{1}{3}}$ OR $\log_{1000} 1000^{\frac{1}{9}}$ OR $3x = \frac{1}{3}$ ***(A1)***

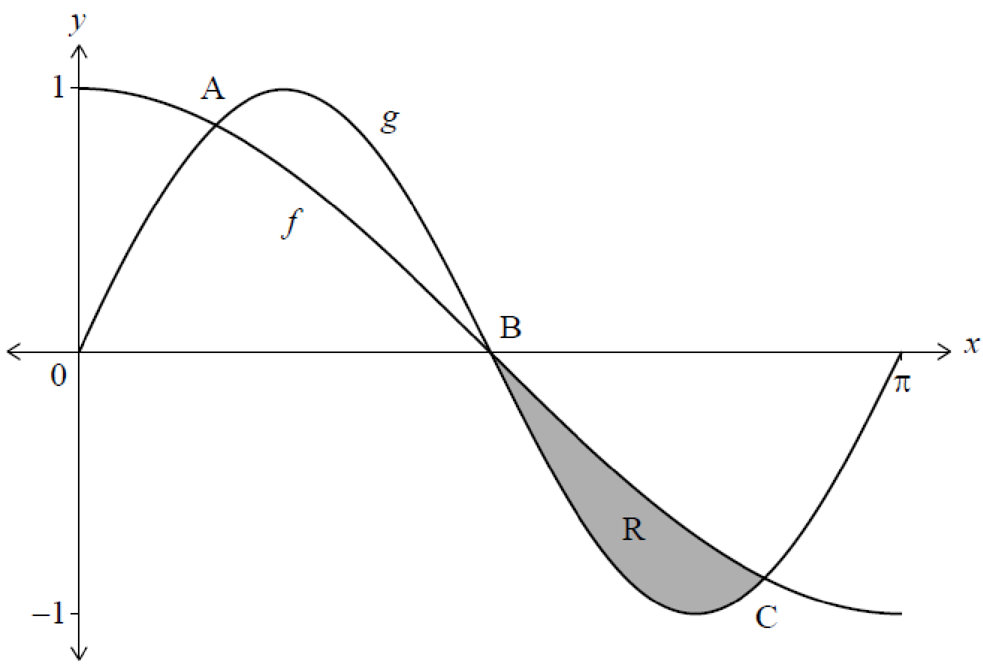
$= \frac{1}{9}$ ***A1***

[3 marks]

12. [Maximum mark: 7] 24M.1.AHL.TZ1.4

Consider the functions $f(x) = \cos x$ and $g(x) = \sin 2x$, where $0 \leq x \leq \pi$.

The graph of f intersects the graph of g at the point A, the point $\mathrm{B}\left(\frac{\pi}{2}, 0\right)$ and the point C as shown on the following diagram.



(a) Find the x-coordinate of point A and the x-coordinate of point C. [3]

Markscheme

recognising $\cos x = 2 \sin x \cos x$ ***(M1)***

$(\cos x \neq 0)$ so $\sin x = \frac{1}{2}$ OR one correct value (accept degrees) ***(A1)***

x-coordinates $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ ***A1***

Note: Award *(M1)(A1)A0* for solutions of 30° and 150°.

[3 marks]

The shaded region R is enclosed by the graph of f and the graph of g between the points B and C.

(b) Find the area of R. [4]

Markscheme

METHOD 1

attempt to integrate $\pm(\cos x - \sin 2x)$ *(M1)*

$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (\cos x - \sin 2x)\ \mathrm{d}\ x$ OR $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (\cos x - 2\sin x \cos x)\ \mathrm{d}\ x$

$= \left[\sin x + \frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ OR $= \left[\sin x - \sin^2 x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ *A1*

Note: Award *A1* for $\pm$ correct integration. Condone incorrect or absent limits up to this point.

attempt to substitute their limits into their integral and subtract *M1*

$= \left(\sin\left(\frac{5\pi}{6}\right) + \frac{1}{2}\cos\left(\frac{5\pi}{3}\right)\right) - \left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\cos(\pi)\right)$ OR
$\left(\sin\left(\frac{5\pi}{6}\right) - \sin^2\left(\frac{5\pi}{6}\right)\right) - \left(\sin\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\pi}{2}\right)\right)$

$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(1 - \frac{1}{2}\right)$ OR $= \left(\frac{1}{2} - \frac{1}{4}\right) - (1 - 1)$

area $= \frac{1}{4}$ *A1*

Note: Award all corresponding marks as appropriate for finding the area between A and B.

Accept work done in degrees.

METHOD 2

$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x \, \mathrm{d}\, x = [\sin x]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ and $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x \, \mathrm{d}\, x = \left[-\frac{1}{2} \cos 2x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ ***A1***

Note: Award ***A1*** for correct integration. Condone incorrect or absent limits up to this point.

attempt to substitute their limits into their integral and subtract (for both integrals) ***M1***

$\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right)$ and $-\frac{1}{2}\cos\left(\frac{5\pi}{3}\right) + \frac{1}{2}\cos(\pi)$

attempt to subtract the two integrals in either order (seen anywhere) ***(M1)***

$\left(\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right)\right) - \left(-\frac{1}{2}\cos\left(\frac{5\pi}{3}\right) + \frac{1}{2}\cos(\pi)\right)$ OR

$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x \, \mathrm{d}\, x - \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x \, \mathrm{d}\, x$

$= \left(\frac{1}{2} - 1\right) - \left(\frac{1}{4} - \frac{1}{2}\right) \ \left(= -\frac{1}{4}\right)$

area $= \frac{1}{4}$ ***A1***

Note: Award all corresponding marks as appropriate for finding the area between A and B.

Accept work done in degrees.

[4 marks]

13. [Maximum mark: 5] 24M.1.AHL.TZ1.5

Consider a geometric sequence with first term 1 and common ratio 10.

S_n is the sum of the first n terms of the sequence.

(a) Find an expression for S_n in the form $\frac{a^n-1}{b}$, where $a,\ b \in \mathbb{Z}^+$. [1]

Markscheme

$S_n = \frac{10^n-1}{9}$ ***A1***

$(a = 10,\ b = 9)$

[1 mark]

(b) Hence, show that $S_1 + S_2 + S_3 + \ldots + S_n = \frac{10(10^n-1)-9n}{81}$. [4]

Markscheme

METHOD 1

$S_1 + S_2 + S_3 + \ldots + S_n$

$= \frac{10-1}{9} + \frac{10^2-1}{9} + \ldots + \frac{10^n-1}{9}$ ***(A1)***

$= \frac{10-1+10^2-1+10^3-1+\ldots+10^n-1}{9}$ OR $\frac{9\left(10-1+10^2-1+10^3-1+\ldots+10^n-1\right)}{81}$

attempt to use geometric series formula on powers of 10, and collect -1's together ***M1***

$10 + 10^2 + 10^3 + \ldots + 10^n = \frac{10(10^n-1)}{10-1}$ and $-1-1-1\ldots = -n$ ***A1***

$= \frac{\frac{10(10^n-1)}{10-1}-n}{9}$ OR $\frac{9\left(\frac{10(10^n-1)}{10-1}\right)-9n}{81}$ ***A1***

Note: Award ***A1*** for any correct intermediate expression.

$= \frac{10(10^n-1)-9n}{81}$ ***AG***

METHOD 2

attempt to create sum using sigma notation with S_n ***M1***

$$\sum_{i=1}^{n} \frac{10^i-1}{9} \quad \left(= \frac{1}{9}\left(\sum_{i=1}^{n} 10^i - \sum_{i=1}^{n} 1\right)\right)$$

$\sum_{i=1}^{n} 10^i = \frac{10(10^n-1)}{9}$ ***A1***

$\sum_{i=1}^{n} 1 = n$ ***A1***

$= \frac{1}{9}\left(\frac{10(10^n-1)}{9} - n\right)$ OR $\frac{1}{9}\left(\frac{10(10^n-1)-9n}{9}\right)$ ***A1***

$= \frac{10(10^n-1)-9n}{81}$ ***AG***

METHOD 3

let $\mathrm{P}(n)$ be the proposition that $S_1 + S_2 + S_3 + \ldots + S_n = \frac{10(10^n-1)-9n}{81}$

considering $\mathrm{P}(1)$:

$\mathrm{LHS} = S_1 = \frac{10^1-1}{9} = 1$ and $\mathrm{RHS} = \frac{10(10^1-1)-9(1)}{81} = 1$ and so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true *i.e.* $S_1 + S_2 + S_3 + \ldots + S_k = \frac{10(10^k-1)-9k}{81}$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

considering $\mathrm{P}(k+1)$:

$S_1 + S_2 + S_3 + \ldots + S_{k+1} = \frac{10(10^k-1)-9k}{81} + \frac{10^{k+1}-1}{9}$

$= \frac{10^{k+1}-10-9k+9(10^{k+1})-9}{81}$ ***A1***

$= \frac{10(10^{k+1}-1)-9(k+1)}{81}$

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true ***R1***

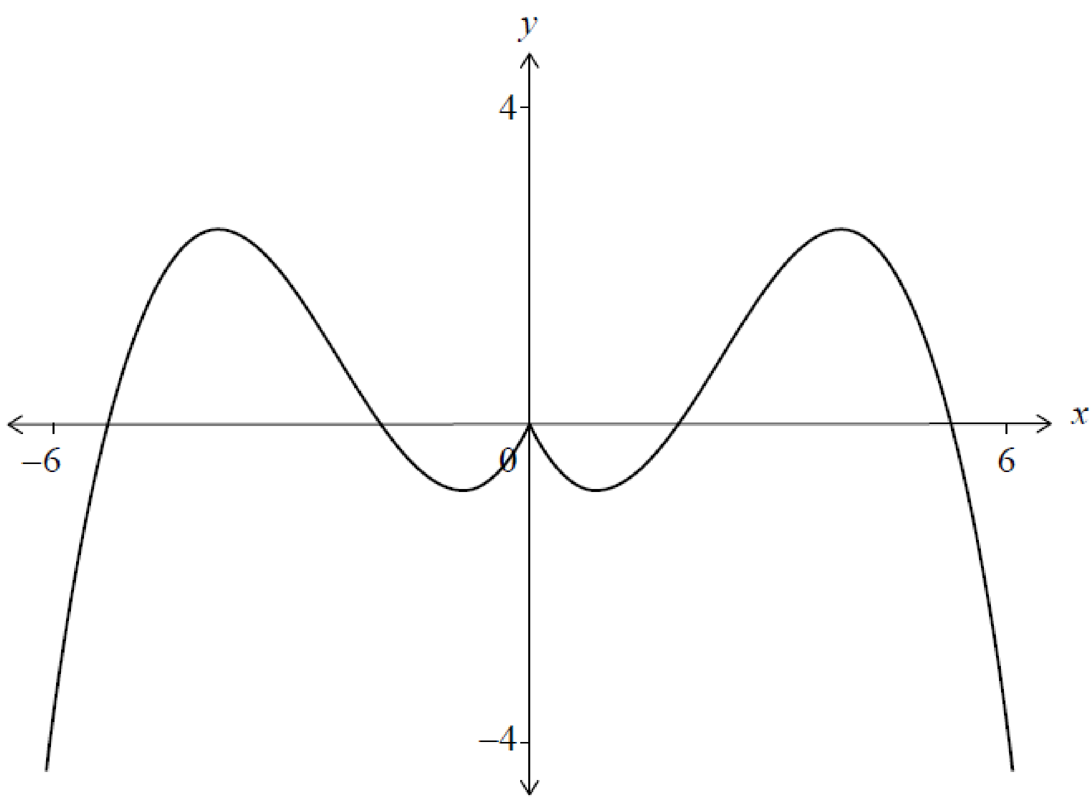
(for all integers $n \geq 1$)

Note: To obtain the final ***R1***, the first ***R1*** and ***A1*** must have been awarded.

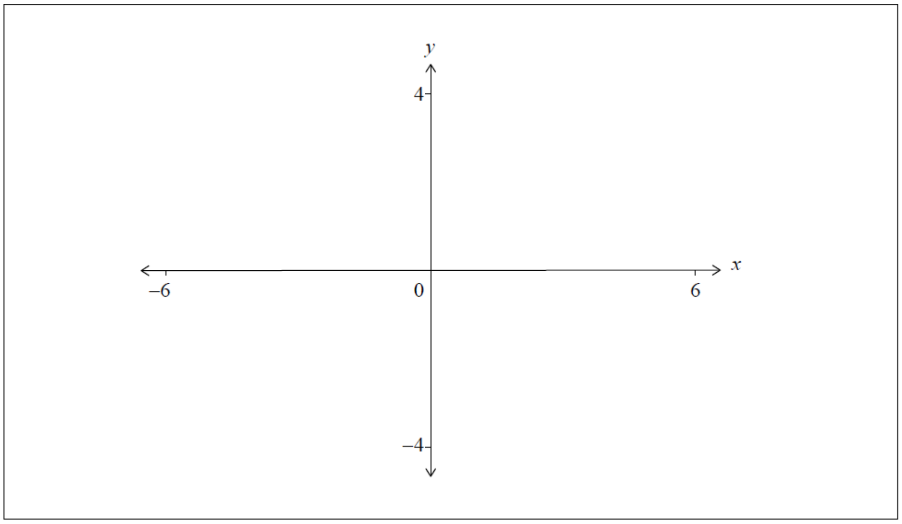
[4 marks]

14. [Maximum mark: 6] 24M.1.AHL.TZ1.9

The graph of $y = f(|x|)$ for $-6 \leq x \leq 6$ is shown in the following diagram.

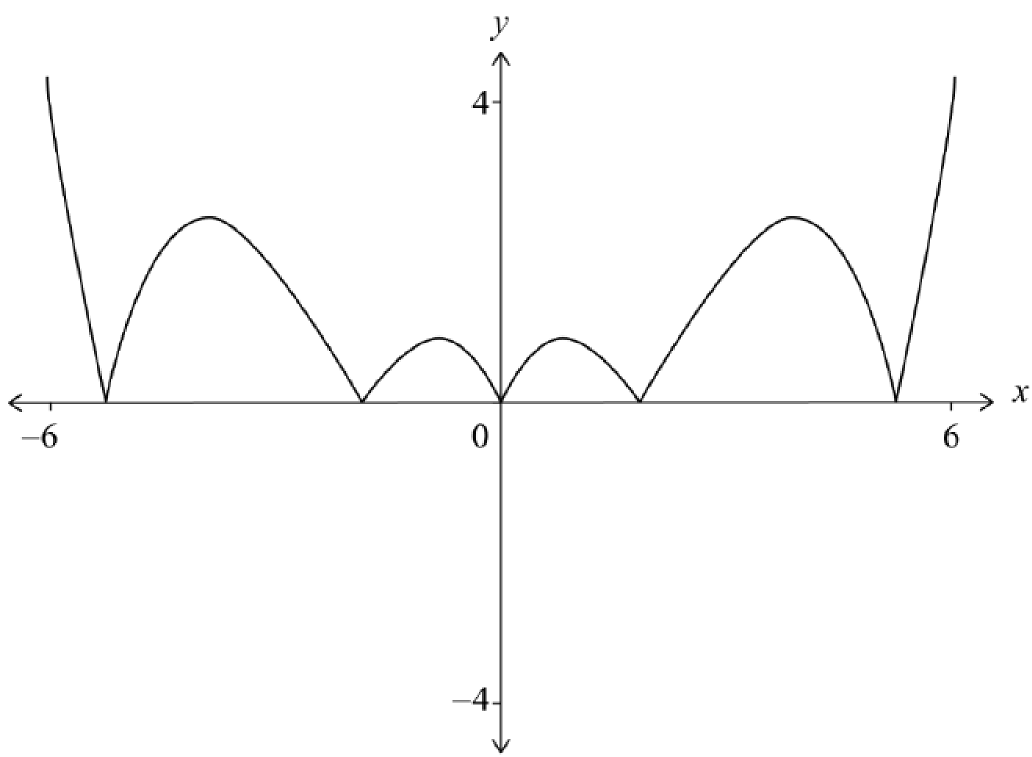


(a) On the following axes, sketch the graph of $y = |f(|x|)|$ for $-6 \leq x \leq 6$.



[2]

Markscheme



reflection of all negative sections in x-axis ***(M1)***

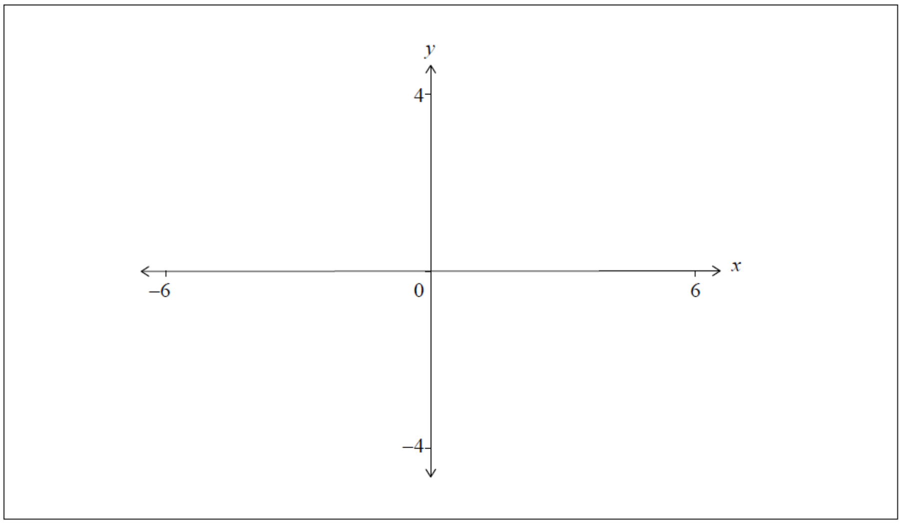
approximately correct graph with sharp points (cusps) at x-intercepts ***A1***

Note: Award ***A1*** only if the heights of the maximum in the middle are lower than the heights of the maximum at the ends.

[2 marks]

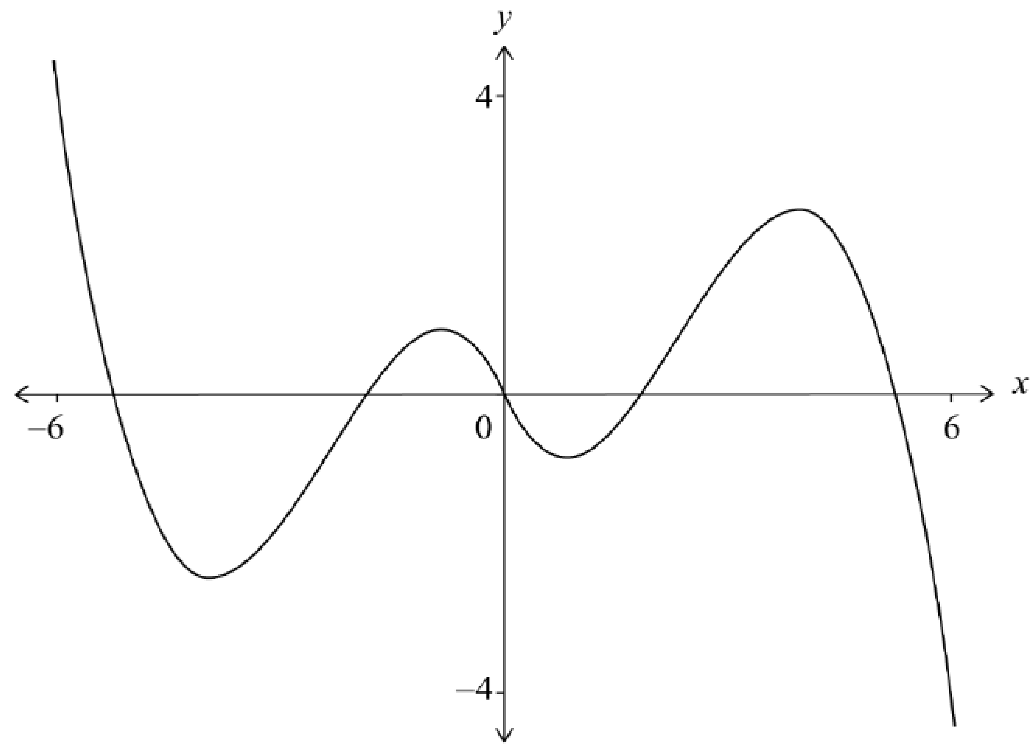
It is given that f is an odd function.

(b) On the following axes, sketch the graph of $y = f(x)$ for $-6 \le x \le 6$.



[2]

Markscheme



A1A1

Note: Award ***A1*** for right hand side unchanged and ***A1*** for rotation 180° about the origin.

[2 marks]

It is also given that $\int_0^4 f(|x|)\ \mathrm{d}\ x = 1.6$.

(c) Write down the value of

(c.i) $\int_{-4}^0 f(x)\ \mathrm{d}\ x$; [1]

Markscheme

-1.6 ***A1***

[1 mark]

(c.ii) $\int_{-4}^4 (f(|x|) + f(x))\ \mathrm{d}\ x$. [1]

Markscheme

3.2 ***A1***

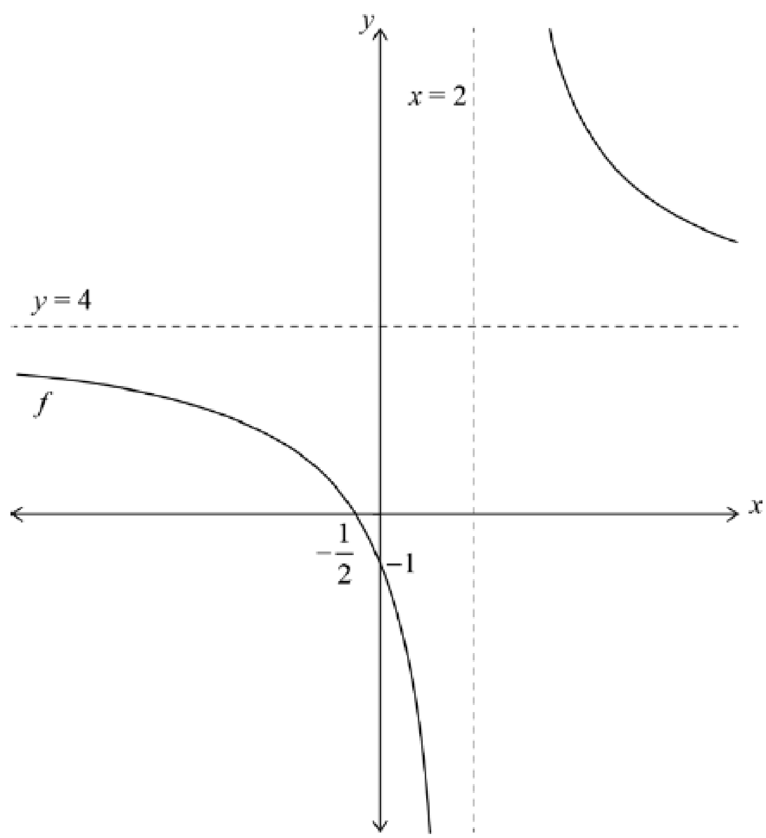
[1 mark]

15. [Maximum mark: 16] 24M.1.AHL.TZ1.10

Consider the function $f(x) = \frac{4x+2}{x-2},\ x \neq 2$.

(a) Sketch the graph of $y = f(x)$. On your sketch, indicate the values of any axis intercepts and label any asymptotes with their equations. [5]

Markscheme



vertical asymptote $x = 2$ sketched and labelled with correct equation ***A1*** horizontal asymptote $y = 4$ sketched and labelled with correct equation ***A1***

For an approximate rational function shape:

labelled intercepts $-\frac{1}{2}$ on x-axis, -1 on y-axis ***A1A1***

two branches in correct opposite quadrants with correct asymptotic behaviour ***A1***

Note: These marks may be awarded independently.

[5 marks]

(b) Write down the range of f. [1]

Markscheme

$y \neq 4$ (or equivalent) ***A1***

[1 mark]

Consider the function $g(x) = x^2 + bx + c$. The graph of g has an axis of symmetry at $x = 2$.

The two roots of $g(x) = 0$ are $-\frac{1}{2}$ and p, where $p \in \mathbb{Q}$.

(c) Show that $p = \frac{9}{2}$. [1]

Markscheme

$2 + \frac{5}{2}$ OR $\left(-\frac{1}{2}\right) + 2 \times \frac{5}{2}$ OR $\frac{-\frac{1}{2}+p}{2} = 2$ OR $-4 = -p + \frac{1}{2}$ ***A1***

$p = \frac{9}{2}$ ***AG***

[1 mark]

(d) Find the value of b and the value of c. [3]

Markscheme

METHOD 1

attempt to substitute both roots to form a quadratic ***(M1)***

EITHER

$\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$ OR $x^2-\left(-\frac{1}{2}+\frac{9}{2}\right)x+\left(-\frac{1}{2}\times\frac{9}{2}\right)$

$=x^2-4x-\frac{9}{4}$ ***A1A1***

$\left(b=-4, c=-\frac{9}{4}\right)$

Note: Award ***A1*** for each correct value. They may be embedded or stated explicitly.

OR

$(2x+1)(2x-9)=4\left(x^2-4x-\frac{9}{4}\right)$

$b=-4, c=-\frac{9}{4}$ ***A1A1***

Note: Award ***A1*** for each correct value. They must be stated explicitly.

METHOD 2

$-\frac{b}{2}=2$ OR $4+b=0\Rightarrow b=-4$ ***A1***

attempt to form a valid equation to find c using their b ***(M1)***

$\left(-\frac{1}{2}\right)^2+-4\left(-\frac{1}{2}\right)+c=0$ OR $\left(\frac{9}{2}\right)^2+-4\left(\frac{9}{2}\right)+c=0$

$c=-\frac{9}{4}$ ***A1***

METHOD 3

attempt to form two valid equations in b and c ***(M1)***

$(-\frac{1}{2})^2 + b(-\frac{1}{2}) + c = 0,\ (\frac{9}{2})^2 + b(\frac{9}{2}) + c = 0$

$b = -4, c = -\frac{9}{4}$ ***A1A1***

METHOD 4

attempt to write $g(x)$ in the form $(x-h)^2 + k$ and substitute for $x,\ h$ and $g(x)$ ***(M1)***

$(-\frac{1}{2} - 2)^2 + k = 0 \Rightarrow k = -\frac{25}{4}$

$(x-2)^2 - \frac{25}{4}$

$= x^2 - 4x - \frac{9}{4}$ ***A1A1***

$(b = -4, c = -\frac{9}{4})$

Note: Award ***A1*** for each correct value. They may be embedded or stated explicitly.

[3 marks]

(e) Find the y-coordinate of the vertex of the graph of $y = g(x)$. [2]

Markscheme

attempt to substitute $x = 2$ into their $g(x)$ OR

complete the square on their $g(x)$ (may be seen in part (d)) ***(M1)***

$y = -\frac{25}{4}$ ***A1***

[2 marks]

(f) Find the product of the solutions of the equation $f(x) = g(x)$. [4]

Markscheme

$\frac{4x+2}{x-2} = \left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right)$ OR $\frac{4x+2}{x-2} = x^2 - 4x - \frac{9}{4}$

attempt to form a cubic equation ***(M1)***

EITHER

$4x + 2 = (x-2)\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right)$ OR $4x + 2 = \left(x^2 - 4x - \frac{9}{4}\right)(x-2)$ OR

$(x-2)\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right) - 4x - 2$ OR $(x-2)\left(x^2 - 4x - \frac{9}{4}\right) - 4x - 2$

$x^3 + \ldots + \frac{5}{2}(=0)$ OR $4x^3 + \ldots + 10(=0)$ ***(A1)(A1)***

Note: Award ***(A1)*** for each of the terms x^3 and $\frac{5}{2}$ or $4x^3$ and 10. Ignore extra terms.

product of roots $= \left(\frac{(-1)^3 \times \frac{5}{2}}{1}\right)$ OR $\left(\frac{(-1)^3 \times 10}{4}\right)$

$= -\frac{5}{2}$ ***A1***

OR

$4\left(x + \frac{1}{2}\right) = (x-2)\left(x + \frac{1}{2}\right)\left(x - \frac{9}{2}\right)$

$x = -\frac{1}{2}$ ***(A1)***

or $4 = x^2 + \ldots + 9 \Rightarrow x^2 + \ldots + 5 = 0$

product of roots of quadratic is 5 ***(A1)***

product is therefore $-\frac{1}{2} \times 5$

$= -\frac{5}{2}$ ***A1***

[4 marks]

16. [Maximum mark: 17] 24M.1.AHL.TZ1.11

Consider the polynomial $P(x) = 3x^3 + 5x^2 + x - 1$.

(a) Show that $(x+1)$ is a factor of $P(x)$. [2]

Markscheme

attempt to substitute -1 into $P(x)$ OR use of synthetic division OR long division ***M1***

$3(-1)^3 + 5(-1)^2 + (-1) - 1 = 0$ OR

	3	5	1	-1
-1		-3	-2	1
	3	2	-1	0

OR Math input error ***A1***

[2 marks]

(b) Hence, express $P(x)$ as a product of three linear factors. [3]

Markscheme

attempt to divide $P(x)$ by $(x+1)$ e.g. using long division or synthetic division ***(M1)***

$P(x) = (x+1)(3x^2 + 2x - 1)$ ***(A1)***

$= (x+1)(x+1)(3x-1)\left(= (x+1)^2(3x-1)\right)$ ***A1***

[3 marks]

Now consider the polynomial $Q(x) = (x+1)(2x+1)$.

(c) Express $\frac{1}{Q(x)}$ in the form $\frac{A}{x+1} + \frac{B}{2x+1}$, where $A,\ B \in \mathbb{Z}$. [3]

Markscheme

$\frac{1}{(x+1)(2x+1)} \equiv \frac{A}{x+1} + \frac{B}{2x+1} \Rightarrow 1 \equiv A(2x+1) + B(x+1)$

attempt to equate both coefficients OR substitute two values eg -1 and $-\frac{1}{2}$ ***(M1)***

$2A+B=0$ and $A+B=1$ OR $1=-A$ and $1=\frac{1}{2}B$

$A=-1$ and $B=2$ ***A1A1***

Note: Award ***A1*** for each value.

$\frac{1}{(x+1)(2x+1)}=\frac{1}{x+1}+\frac{2}{2x+1}$

[3 marks]

(d) Hence, or otherwise, show that $\frac{1}{(x+1)Q(x)}=\frac{4}{2x+1}-\frac{2}{x+1}-\frac{1}{(x+1)^2}$. [2]

Markscheme

$\frac{1}{(x+1)(x+1)(2x+1)}$

$=\frac{1}{(x+1)}\left(-\frac{1}{x+1}+\frac{2}{2x+1}\right)$ ***(A1)***

$=-\frac{1}{(x+1)^2}+\frac{2}{(2x+1)(x+1)}\left(=-\frac{1}{(x+1)^2}+2\left(-\frac{1}{x+1}+\frac{2}{2x+1}\right)\right)$ ***A1***

$=\frac{4}{2x+1}-\frac{2}{x+1}-\frac{1}{(x+1)^2}$ ***AG***

Note: Award ***A1A0*** for follow through from incorrect values in part (c).

[2 marks]

(e) Hence, find $\int\frac{1}{(x+1)^2(2x+1)}\,\mathrm{d}\,x$. [4]

Markscheme

attempt to integrate at least one term in $\left(\frac{4}{2x+1}-\frac{2}{x+1}-\frac{1}{(x+1)^2}\right)$ ***(M1)***

$\int\left(\frac{4}{2x+1}-\frac{2}{x+1}-\frac{1}{(x+1)^2}\right)\mathrm{d}\,x$

$= 2 \ln |2x+1| - 2 \ln |x+1| + \frac{1}{x+1} (+c)$ ***A1A1***

Note: Award ***A1*** for each correct term.

Award a maximum of ***M1A1A0A1*** if modulus signs are omitted.

Condone the absence of $+c$.

[4 marks]

Consider the function defined by $f(x) = \frac{P(x)}{(x+1)Q(x)}$, where $x \neq -1,\ x \neq -\frac{1}{2}$.

(f) Find

(f.i) $\lim_{x \to -1} f(x)$; [2]

Markscheme

METHOD 1

attempt to cancel factors and substitute $x = -1$ ***(M1)***

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \left(\frac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)} \right) = \lim_{x \to -1} \left(\frac{3x-1}{2x+1} \right) = \frac{3(-1)-1}{2(-1)+1}$$

$= 4$ ***A1***

METHOD 2

attempt to expand denominator, differentiate numerator and denominator twice and substitute $x = -1$ ***(M1)***

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \left(\frac{3x^3+5x^2+x-1}{2x^3+5x^2+4x+1} \right) = \lim_{x \to -1} \left(\frac{9x^2+10x+1}{6x^2+10x+4} \right) = \lim_{x \to -1} \left(\frac{18x+10}{12x+10} \right) = \frac{18(-1)+10}{12(-1)+10}$$

$= 4$ ***A1***

[2 marks]

(f.ii) $\lim_{x \to \infty} f(x)$. [1]

Markscheme

METHOD 1

attempt to consider coefficients of x^3 or divide all terms by x^3

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(\frac{3x^3 + \ldots}{2x^3 + \ldots} \right)$ or $\lim_{x \to \infty} \left(\frac{3 + \text{terms which tend to } 0}{2 + \text{terms which tend to } 0} \right)$

$= \frac{3}{2}$ ***A1***

METHOD 2

attempt to cancel factors and consider coefficients of x or divide all terms by x

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(\frac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)} \right) = \lim_{x \to \infty} \left(\frac{3x-1}{2x+1} \right)$ or $\lim_{x \to \infty} \left(\frac{3 - \frac{1}{x}}{2 + \frac{1}{x}} \right)$

$= \frac{3}{2}$ ***A1***

METHOD 3

attempt to expand denominator, differentiate numerator and denominator three times

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(\frac{3x^3+5x^2+x-1}{2x^3+5x^2+4x+1} \right) = \lim_{x \to \infty} \left(\frac{9x^2+10x+1}{6x^2+10x+4} \right) = \lim_{x \to \infty} \left(\frac{18x+10}{12x+10} \right) = \lim_{x \to \infty} \left(\frac{18}{12} \right)$

$= \frac{3}{2}$ ***A1***

Note: If the ***M1*** has not been awarded in part (i) it can be awarded in part (ii).

[3 marks]

17. [Maximum mark: 4] 24M.1.AHL.TZ2.1

Solve $\tan(2x - 5^\circ) = 1$ for $0^\circ \le x \le 180^\circ$. [4]

Markscheme

$\tan^{-1} 1 = 45^\circ$ or equivalent ***(A1)***

attempt to equate $2x - 5^\circ$ to their reference angle ***(M1)***

Note: Do not accept $2x - 5^\circ = 1$.

$2x - 5^\circ = 45^\circ, (225^\circ)$

$x = 25^\circ, 115^\circ$ ***A1A1***

Note: Do not award the final ***A1*** if any additional solutions are seen.

[4 marks]

18. [Maximum mark: 7] 24M.1.AHL.TZ2.7

A function $g(x)$ is defined by $g(x) = 2x^3 - 7x^2 + dx - e$, where $d,\ e \in \mathbb{R}$.

$\alpha,\ \beta$ and γ are the three roots of the equation $g(x) = 0$ where $\alpha,\ \beta,\ \gamma \in \mathbb{R}$.

(a) Write down the value of $\alpha + \beta + \gamma$. [1]

Markscheme

$\alpha + \beta + \gamma = \frac{7}{2}$ ***A1***

[1 mark]

A function $h(z)$ is defined by $h(z) = 2z^5 - 11z^4 + rz^3 + sz^2 + tz - 20$, where $r,\ s,\ t \in \mathbb{R}$.

$\alpha,\ \beta$ and γ are also roots of the equation $h(z) = 0$.

It is given that $h(z) = 0$ is satisfied by the complex number $z = p + 3\text{i}$.

(b) Show that $p = 1$. [3]

Markscheme

$p - 3\text{i}$ is also a root (seen anywhere) ***A1***

recognition of 5 roots and attempt to sum these roots ***(M1)***

$p + 3\text{i} + p - 3\text{i} + \frac{7}{2}$

$p + 3\text{i} + p - 3\text{i} + \frac{7}{2} = \frac{11}{2}$ ***A1***

$p = 1$ ***AG***

[3 marks]

It is now given that $h\left(\frac{1}{2}\right) = 0$, and $\alpha,\ \beta \in \mathbb{Z}^+,\ \alpha < \beta$ and $\gamma \in \mathbb{Q}$.

(c.i) Find the value of the product $\alpha\beta$. [2]

Markscheme

attempt to find product of 5 roots and equate to ± 10 ***(M1)***

$(1 + 3\text{i})(1 - 3\text{i})\frac{1}{2}\alpha\beta = 10$

$\alpha\beta = 2$ ***A1***

[2 marks]

(c.ii) Write down the value of α and the value of β. [1]

Markscheme

$\alpha = 1$ and $\beta = 2$ ***A1***

[1 mark]

19. [Maximum mark: 6] 24M.1.AHL.TZ2.8

Use l'Hôpital's rule to find $\lim_{x \to 0} \frac{\sec^4 x - \cos^2 x}{x^4 - x^2}$. [6]

Markscheme

Note: To award full marks limit notation $\lim_{x \to 0}$ must be seen at least once in their working. If no limit notation is seen but otherwise all correct, do not award the final ***A1***.

$\lim_{x \to 0} \frac{4 \sec^2 x \tan x + 2 \sin x \cos x}{4x^3 - 2x}$ ***A1A1***

Note: Award ***A1*** for numerator and ***A1*** for denominator.

$= \lim_{x \to 0} \frac{16 sec^4 x \tan^2 x + 4 sec^6 x - 2 \sin^2 x + 2 \cos^2 x}{12x^2 - 2}$ ***M1A1A1***

Note: Award ***M1 for second use of*** l'Hôpital's rule providing their expression is in indeterminate form as $x \to 0$ and providing there is **no third attempt** at using l'Hôpital's Rule.

$= -3$ ***A1***

[6 marks]

20. [Maximum mark: 16] 24M.1.AHL.TZ2.10

Consider the arithmetic sequence $a,\ p,\ q \ldots$, where $a,\ p,\ q \neq 0$.

(a) Show that $2p - q = a$. [2]

Markscheme

attempt to find a difference ***(M1)***

$d = p - a, 2d = q - a, d = q - p$ OR $p = a + d, q = a + 2d, q = p + d$

correct equation ***A1***

$p - a = q - p$ OR $q - a = 2(p - a)$ OR $p = \frac{a+q}{2}$ (or equivalent)

$2p - q = a$ ***AG***

[2 marks]

Consider the geometric sequence $a,\ s,\ t\ldots$, where $a,\ s,\ t \neq 0$.

(b) Show that $s^2 = at$. [2]

Markscheme

attempt to find a ratio ***(M1)***

$r = \frac{s}{a},\ r^2 = \frac{t}{a},\ r = \frac{t}{s}$ OR $s = ar, t = ar^2, t = sr$

correct equation ***A1***

$\left(\frac{s}{a}\right)^2 = \frac{t}{a}$ OR $\frac{s}{a} = \frac{t}{s}$ (or equivalent)

$s^2 = at$ ***AG***

[2 marks]

The first term of both sequences is a.

It is given that $q = t = 1$.

(c) Show that $p > \frac{1}{2}$. [2]

Markscheme

EITHER

$2p - 1 = s^2$ (or equivalent) ***A1***

$(s^2 > 0) \Rightarrow 2p - 1 > 0$ OR $s = \sqrt{2p-1} \Rightarrow 2p - 1 > 0$ OR $p = \frac{s^2+1}{2}$ (and $s^2 > 0$) ***R1***

OR

$2p - 1 = a$ and $s^2 = a$ ***A1***

($s^2 > 0$, so) $a > 0 \Rightarrow 2p - 1 > 0$ OR $p\frac{a+1}{2}$ and $a > 0$ ***R1***

$\Rightarrow p > \frac{1}{2}$ ***AG***

Note: Do not award ***A0R1***.

[2 marks]

Consider the case where $a = 9,\ s > 0$ and $q = t = 1$.

(d) Write down the first four terms of the

(d.i) arithmetic sequence; [2]

Markscheme

$9, 5, 1, -3$ ***A1A1***

Note: Award ***A1*** for each of 2nd term and 4th term

[2 marks]

(d.ii) geometric sequence. [2]

Markscheme

$9, 3, 1, \frac{1}{3}$ ***A1A1***

Note: Award ***A1*** for each of 2nd term and 4th term

[2 marks]

The arithmetic and the geometric sequence are used to form a new arithmetic sequence u_n.

The first three terms of u_n are $u_1 = 9 + \ln 9,\ u_2 = 5 + \ln 3$, and $u_3 = 1 + \ln 1$.

(e.i) Find the common difference of the new sequence in terms of $\ln 3$. [3]

Markscheme

attempt to find the difference between two consecutive terms ***(M1)***

$d = u_2 - u_1 = 5 + \ln 3 - 9 - \ln 9$ OR $d = u_3 - u_2 = 1 + \ln 1 - 5 - \ln 3$

$\ln 9 = 2 \ln 3$ OR $\ln 1 = 0$ OR $\ln 3 - \ln 9 = \ln \frac{1}{3} \left(= \ln 3^{-1} = -\ln 3\right)$ (seen anywhere) ***(A1)***

$d = -4 - \ln 3$ ***A1***

[3 marks]

(e.ii) Show that $\sum_{i=1}^{10} = -90 - 25 \ln 3$. [3]

Markscheme

METHOD 1

attempt to substitute first term and their common difference into S_{10} ***(M1)***

$\frac{10}{2}(2(9 + \ln 9) + 9(-4 - \ln 3))$ OR $\frac{10}{2}(2(9 + 2 \ln 3) + 9(-4 - \ln 3))$ (or equivalent) ***A1***

$= 5(-18 - 5 \ln 3)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i = -90 - 25 \ln 3$ ***AG***

METHOD 2

$u_{10} = 9 + \ln 9 + 9(-4 - \ln 3)(= -27 + \ln 9 - 9 \ln 3)$

attempt to substitute first term and their u_{10} into S_{10} ***(M1)***

$\frac{10}{2}(2(9 + \ln 9) + 9(-4 - \ln 3))$ OR $\frac{10}{2}(9 + \ln 9 - 27 + \ln 9 - 9 \ln 3)$ OR

$\frac{10}{2}(2(9 + 2 \ln 3) + 9(-4 - \ln 3))$ OR $\frac{10}{2}(9 + \ln 9 - 27 - 7 \ln 3)$ (or equivalent) ***A1***

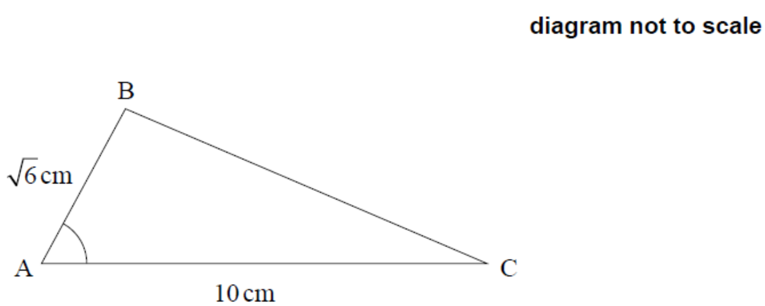
$= 5(-18 - 5 \ln 3)$ (or equivalent in terms of $\ln 3$) ***A1***

$\sum_{i=1}^{10} u_i = -90 - 25 \ln 3$ ***AG***

[3 marks]

21. [Maximum mark: 6] 23N.1.AHL.TZ1.4

In the following triangle ABC, $\mathrm{AB} = \sqrt{6 \text{ cm}}$, $\mathrm{AC} = 10 \text{ cm}$ and $\cos \mathrm{B\hat{A}C} = \frac{1}{5}$.



Find the area of triangle ABC. [6]

Markscheme

METHOD 1

EITHER

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\left(\sqrt{5^2 - 1^2} =\right)\sqrt{24}$ *(A1)*

OR

attempt to use the Pythagorean identity $\cos^2 \alpha + \sin^2 \alpha = 1$ *(M1)*

$\sin^2 \mathrm{B\hat{A}C} = 1 - \left(\frac{1}{5}\right)^2$ *(A1)*

THEN

$\sin \mathrm{B\hat{A}C} = \frac{\sqrt{24}}{5}$ (may be seen in area formula) *A1*

attempt to use 'Area$= \frac{1}{2} ab \sin C$' (must include their calculated value of $\sin \mathrm{B\hat{A}C}$) *(M1)*

$= \frac{1}{2} \times 10 \times \sqrt{6} \times \frac{\sqrt{24}}{5}$ *(A1)*

$= 12$ (cm^2) *A1*

[6 marks]

METHOD 2

attempt to find perpendicular height of triangle BAC *(M1)*

EITHER

$\text{height} = \sqrt{6} \times \sin \mathrm{B\hat{A}C}$

attempt to use the Pythagorean identity $\cos^2 \alpha + \sin^2 \alpha = 1$ *(M1)*

$\text{height} = \sqrt{6} \times \sqrt{1 - \left(\frac{1}{5}\right)^2}$ **(A1)**

$= \sqrt{6} \times \frac{\sqrt{24}}{5} \left(= \frac{12}{5}\right)$ *A1*

OR

$\text{adjacent} = \frac{\sqrt{6}}{5}$ *(A1)*

attempt to use Pythagoras' theorem in a right-angled triangle. *(M1)*

$\text{height} = \sqrt{6 - \frac{6}{25}} \left(= \frac{12}{5}\right)$ (may be seen in area formula) *(A1)*

THEN

attempt to use 'Area$=\frac{1}{2}\text{base}\times\text{height}$' (must include their calculated value for height) ***(M1)***

$=\frac{1}{2}\times 10\times\frac{12}{5}$

$=12$ (cm^2) ***A1***

[6 marks]

22. [Maximum mark: 5] 23N.1.AHL.TZ1.7

It is given that $z=5+q\text{i}$ satisfies the equation $z^2+\text{i}z=-p+25\text{i}$, where $p,\ q\ \in\mathbb{R}$.

Find the value of p and the value of q. [5]

Markscheme

METHOD 1

attempt to substitute solution into given equation ***(M1)***

$(5+q\text{i})^2+\text{i}(5+q\text{i})=-p+25\text{i}$

$25-q^2+10q\text{i}-q+5\text{i}+p-25\text{i}=0$ OR $25-q^2+10q\text{i}-q+5\text{i}=-p+25\text{i}$ ***A1***

$25-q^2+p-q+(10q-20)\text{i}=0$

attempt to equate real or imaginary parts: ***(M1)***

$10q-20=0$ OR $25-q^2+p-q=0$

$q=2,\ p=-19$ ***A1A1***

METHOD 2

$z^2+\text{i}z+p-25\text{i}=0$

sum of roots $= -\mathrm{i}$, product of roots $= p - 25\mathrm{i}$ ***M1***

one root is $(5 + q\mathrm{i})$ so other root is $(-5 - q\mathrm{i} - \mathrm{i})$ ***A1***

product $(5 + q\mathrm{i})(-5 - q\mathrm{i} - \mathrm{i}) = -25 - 5q\mathrm{i} - 5\mathrm{i} - 5q\mathrm{i} + q^2 + q = p - 25\mathrm{i}$

equating real and imaginary parts for product of roots ***(M1)***

Im:$-25 = 10q - 5$ Re: $p = -25 + q^2 + q$

$q = 2,\ p = -19$ ***A1A1***

[5 marks]

23. [Maximum mark: 9] 23N.1.AHL.TZ1.8

(a) Find $\int x\ (\ln\ x)^2\ \mathrm{d}x$ [6]

Markscheme

METHOD 1

attempt to integrate by parts ***(M1)***

$u = (\ln\ x)^2, \mathrm{d}v = x\mathrm{d}x$ ***(M1)***

$\int x(\ln\ x)^2\ \mathrm{d}x = \frac{x^2(\ln\ x)^2}{2} - \int x\ \ln\ x\ \mathrm{d}x$ ***A1***

attempt to integrate $x\ \ln\ x$ by parts, with $u = \ \ln\ x$ ***(M1)***

$\int x\ \ln\ x\ \mathrm{d}x = \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \mathrm{d}x\right]$ ***A1***

$\int\ x\ (\ln\ x)^2\ \mathrm{d}x = \frac{x^2(\ln\ x)^2}{2} - \left[\frac{x^2\ \ln\ x}{2} - \int \frac{x}{2}\ \mathrm{d}x\right]$

$= \frac{x^2(\ln\ x)^2}{2} - \frac{x^2\ \ln\ x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

METHOD 2 (knowing $\int\ \ln\ x\ \mathrm{d}x\ =\ x\ \ln x\ -\ x$)

attempt to integrate by parts ***(M1)***

$u = x \ln x, \mathrm{d}v = \ln x\mathrm{d}x$ ***(M1)***

$\int x \ln x (\ln x) \, \mathrm{d}x = x \ln x \, (x \ln x - x) - \int (\ln x + 1)(x \ln x - x)\mathrm{d}x$
A1

$= x \ln x \, (x \ln x - x) - \int x \, (\ln x)^2 \mathrm{d}x + \int x\mathrm{d}x$ ***A1***

$2I = x \ln x \, (x \ln x - x) + \frac{x^2}{2} + c$ ***M1***

$I = \frac{x^2 (\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} (+c)$ ***A1***

[6 marks]

METHOD 3 (knowing $\int x \ln x \, \mathrm{d}x$)

$\int x \ln x \mathrm{d}x = \frac{x^2 (\ln x)}{2} - \frac{x^2}{4}$

attempt to integrate by parts ***(M1)***

$u = \ln x, \ \mathrm{d}v = x \ln x\mathrm{d}x$ ***(M1)***

$\int (x \ln x) \ln x \, \mathrm{d}x = \ln x \left(\frac{x^2 (\ln x)}{2} - \frac{x^2}{4} \right) - \int \frac{1}{x} \left(\frac{x^2 (\ln x)}{2} - \frac{x^2}{4} \right) \mathrm{d}x$ ***A1***

$= \ln x \left(\frac{x^2 (\ln x)}{2} - \frac{x^2}{4} \right) - \int \left(\frac{x (\ln x)}{2} - \frac{x}{4} \right) \mathrm{d}x$ ***A1***

$= \ln x \left(\frac{x^2 (\ln x)}{2} - \frac{x^2}{4} \right) - \frac{x^2 \ln x}{4} + \frac{x^2}{8} + \frac{x^2}{8}$ ***A1***

$= \frac{x^2 (\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} (+c)$ ***A1***

[6 marks]

(b) Hence, show that $\int_1^4 x (\ln x)^2 \mathrm{d}x = 32 (\ln 2)^2 - 16 \ln 2 + \frac{15}{4}$. [3]

Markscheme

attempt to substitute limits into their integrated expression ***(M1)***

$\left[\frac{x^2 \, (\ln x^2)}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} \right]_1^4 = \left(8 (\ln 4)^2 - 8 \ln 4 + 4 \right) - \left(\frac{1}{4} \right)$

attempt to replace any $\ln 4$ term with $2 \ln 2$ ***(M1)***

$= 8(2\ln 2)^2 - 8(2\ln 2) + 4 - \frac{1}{4}$ ***A1***

$= 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$ ***AG***

[3 marks]

24. [Maximum mark: 8] 23N.1.AHL.TZ1.9

Consider the function $f(x) = \frac{\sin^2(kx)}{x^2}$, where $x \neq 0$ and $k \in \mathbb{R}^+$.

(a) Show that f is an even function. [2]

Markscheme

$f(-x) = \frac{\sin^2(-kx)}{(-x)^2}$ ***M1***

$= \frac{(-\sin(kx))^2}{(-x)^2}$ ***A1***

$= \frac{\sin^2(kx)}{x^2} = (f(x))$

hence $f(x)$ is even ***AG***

[2 marks]

(b) Given that $\lim_{x\to 0} f(x) = 16$, find the value of k. [6]

Markscheme

METHOD 1

Noting that $\lim_{x\to 0}(f(x)) = \frac{0}{0}$ ***(M1)***

attempt to differentiate numerator and denominator: ***M1***

$\lim_{x\to 0}(f(x)) = \lim_{x\to 0}\left(\frac{2k\sin kx\cos kx}{2x}\right)\left(= \lim_{x\to 0}\left(\frac{k\sin 2kx}{2x}\right)\right)$ ***A1***

(evaluates to $\frac{0}{0}$) and attempts to differentiate a second time: ***M1***

$$= \lim_{x \to 0}\left(\frac{2k^2(\cos^2 kx - \sin^2 kx)}{2}\right) \left(= \lim_{x \to 0}\left(\frac{2k^2 \cos 2kx}{2}\right)\right) = k^2$$ ***A1***

$(k^2 = 16 \Rightarrow) k = 4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x \to 0}$'is not explicitly seen.

METHOD 2

attempt to express $\sin (kx)$ as a Maclaurin series ***M1***

$\sin (kx) = kx\ (+\ldots)$

$\sin^2 (kx) = k^2x^2\ (+\ldots)$ ***A1***

$$\lim_{x \to 0}\left(\frac{\sin^2 (kx)}{x^2}\right) = \lim_{x \to 0}\left(\frac{k^2x^2\ (+\ldots)}{x^2}\right)$$ ***M1***

$$= \lim_{x \to 0}\left(k^2 + \text{terms in } x\right)$$ ***R1***

Note: This ***R1*** is awarded independently of any other marks.

$= k^2$ ***A1***

$(k^2 = 16 \Rightarrow) k = 4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x \to 0}$'is not explicitly seen.

METHOD 3

splitting function into $\left(\frac{\sin kx}{x}\right)\left(\frac{\sin kx}{x}\right)$ and using limit of product = product of limits ***(M1)***

$$\lim_{x \to 0}\left(\frac{\sin^2 (kx)}{x^2}\right) = \lim_{x \to 0}\left(\frac{\sin (kx)}{x}\right) \times \lim_{x \to 0}\left(\frac{\sin kx}{x}\right)$$ ***A1***

EITHER

using L' Hôpitals rule ***(M1)***

$\lim_{x \to 0}\left(\frac{\sin\ kx}{x}\right) = \lim_{x \to 0}\left(\frac{k\ \cos\ kx}{1}\right) = k$ ***(A1)***

OR

using Maclaurin expansion for $\sin\ kx$ ***(M1)***

$\sin\ (kx) = kx\ (+\ldots)$

$\lim_{x \to 0}\left(\frac{\sin\ (kx)}{x}\right) = \lim_{x \to 0}\ \left(\frac{kx+\ldots}{x}\right) = \lim_{x \to 0}(k + \text{terms in } x) = k$ ***(A1)***

THEN

hence $\lim_{x \to 0}\ \left(\frac{\sin^2\ (kx)}{x^2}\right) = k \times k = k^2$ ***A1***

$k^2 = 16\ \Rightarrow k = 4(k > 0)$ ***A1***

Note: Award relevant marks, even if '$\lim_{x \to 0}$'is not explicitly seen.

[6 marks]

25. [Maximum mark: 15] 23N.1.AHL.TZ1.10

The functions f and g are defined by

$f(x) = \ln\ (2x - 9)$, where $x > \frac{9}{2}$

$g(x) = 2\ \ln\ x - \ln\ d$, where $x > 0,\ d \in \mathbb{R}^+$.

(a) State the equation of the vertical asymptote to the graph of $y = g(x)$. [1]

Markscheme

$x = 0$ ***A1***

[1 mark]

The graphs of $y = f(x)$ and $y = g(x)$ intersect at two distinct points.

(b.i) Show that, at the points of intersection, $x^2 - 2dx + 9d = 0$. [4]

Markscheme

setting $\ln(2x - 9) = 2 \ln x - \ln d$ ***M1***

attempt to use power rule ***(M1)***

$2 \ln x = \ln x^2$ (seen anywhere)

attempt to use product/quotient rule for logs ***(M1)***

$\ln(2x - 9) = \ln \frac{x^2}{d}$ OR $\ln \frac{x^2}{2x-9} = \ln d$ OR $\ln(2x - 9)d = \ln x^2$

$\frac{x^2}{d} = 2x - 9$ OR $\frac{x^2}{2x-9} = d$ OR $(2x - 9)d = x^2$ ***A1***

$x^2 - 2dx + 9d = 0$ ***AG***

[4 marks]

(b.ii) Hence show that $d^2 - 9d > 0$. [3]

Markscheme

discriminant $= (-2d)^2 - 4 \times 1 \times 9d$ ***(A1)***

recognizing discriminant > 0 ***(M1)***

$(-2d)^2 - 4 \times 1 \times 9d > 0$ OR $(2d)^2 - 4 \times 9d > 0$ OR $4d^2 - 36d > 0$ ***A1***

$d^2 - 9d > 0$ ***AG***

[3 marks]

(b.iii) Find the range of possible values of d. [2]

Markscheme

setting $d(d - 9) > 0$ OR $d(d - 9) = 0$ OR sketch graph

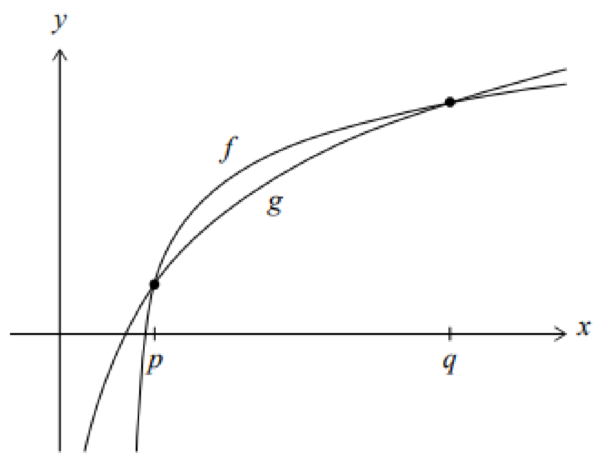
OR sign test OR $d^2 > 9d$ ***(M1)***

$d < 0$ or $d > 9$, but $d \in \mathbb{R}^+$

$d > 9$ (or $]9, \infty[$) ***A1***

[2 marks]

The following diagram shows part of the graphs of $y = f(x)$ and $y = g(x)$.



The graphs intersect at $x = p$ and $x = q$, where $p < q$.

(c) In the case where $d = 10$. find the value of $q - p$. Express your answer in the form of $a\sqrt{b}$, where, $a,\ b \in \mathbb{Z}^+$. [5]

Markscheme

$x^2 - 20x + 90\ (= 0)$ ***A1***

attempting to solve their 3 term quadratic equation ***(M1)***

$\left((x - 10)^2 - 10 = 0\right)$ or $\left((x =)\frac{20 \pm \sqrt{(-20)^2 - 4 \times 1 \times 90}}{2}\right)$

$x = 10 - \sqrt{10}(= p)$ or $x = 10 + \sqrt{10}(= q)$ ***(A1)***

subtracting their values of x ***(M1)***

distance $= 2\sqrt{10}$ **A1**

$(a = 2,\ b = 10)$

Note: Accept $1\sqrt{40}$ OR $\sqrt{40}$.

[5 marks]

26. [Maximum mark: 6] 23M.1.AHL.TZ1.3

Solve $\cos 2x = \sin x$, where $-\pi \leq x \leq \pi$. [6]

Markscheme

$1 - 2\sin^2 x = \sin x$ ***A1***

$2\sin^2 x + \sin x - 1 = 0$

valid attempt to solve quadratic ***(M1)***

$(2\sin x - 1)(\sin x + 1)$ OR $\frac{-1 \pm \sqrt{1 - 4(2)(-1)}}{2(2)}$

recognition to solve for $\sin x$ ***(M1)***

$\sin x = \frac{1}{2}$ OR $\sin x = -1$

any correct solution from $\sin x = -1$ ***A1***

any correct solution from $\sin x = \frac{1}{2}$ ***A1***

Note: The previous two marks may be awarded for degree or radian values, irrespective of domain.

$x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ ***A1***

Note: If no working shown, award no marks for a final value(s).

Award ***A0*** for $-\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$ if additional values also given.

[6 marks]

27. [Maximum mark: 5] 23M.1.AHL.TZ1.6

The side lengths, x cm, of an equilateral triangle are increasing at a rate of $4\ \text{cm s}^{-1}$.

Find the rate at which the area of the triangle, $A\ \text{cm}^2$, is increasing when the side lengths are $5\sqrt{3}$ cm.

[5]

Markscheme

$A = \frac{1}{2}x^2 \sin\frac{\pi}{3}$ OR $A = \frac{1}{2}x^2 \sin 60^\circ$ OR triangle height $h = \sqrt{x^2 - \left(\frac{x}{2}\right)^2}\left(= \frac{\sqrt{3}}{2}x\right)$ ***(A1)***

$= \frac{1}{2}x^2\left(\frac{\sqrt{3}}{2}\right)$ OR $A = \frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right)\left(= \frac{\sqrt{3}}{4}x^2\right)$ ***A1***

Note: Award ***A1*** for $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. This may be seen at a later stage.

attempt to use chain rule or implicit differentiation ***(M1)***

$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2x\frac{dx}{dt}$ OR $\frac{dA}{dt} = \frac{1}{2} \times \sin\frac{\pi}{3} \times 2x\frac{dx}{dt}$ ***(A1)***

$= \frac{2\sqrt{3}}{4} \times 5\sqrt{3} \times 4$

$\frac{dA}{dt} = 30\left(\text{cm}^2\ \text{s}^{-1}\right)$ ***A1***

Note: Award a maximum of ***(A1)A1(M1)(A0)A1*** for a correct answer with incorrect derivative notation seen throughout.

[5 marks]

28. [Maximum mark: 6] 23M.1.AHL.TZ1.7

Consider $P(z) = 4m - mz + \frac{36}{m}z^2 - z^3$, where $z \in \mathbb{C}$ and $m \in \mathbb{R}^+$.

Given that $z - 3\mathrm{i}$ is a factor of $P(z)$, find the roots of $P(z) = 0$. [6]

Markscheme

METHOD 1

$3\mathrm{i}$ (is a root) ***A1***

(other complex root is) $-3\mathrm{i}$ ***A1***

Note: Award ***A1A1*** for $P(3\mathrm{i})$ and $P(-3\mathrm{i}) = 0$ seen in their working.

Award ***A1*** for each correct root seen in sum or product of their roots.

EITHER

attempt to find $P(3\mathrm{i}) = 0$ or $P(-3\mathrm{i}) = 0$ ***(M1)***

$4m - 3m\mathrm{i} + \frac{36}{m}(3\mathrm{i})^2 - (3\mathrm{i})^3 = 0$

$4m - 3m\mathrm{i} - \frac{36}{m}(-9) + 27\mathrm{i} = 0$

attempt to equate the real or imaginary parts ***(M1)***

$27 - 3m = 0$ OR $9 \times \frac{36}{m} = 4m$

OR

attempt to equate sum of three roots to $\frac{36}{m}$ ***(M1)***

Note: Accept sum of three roots set to $-\frac{36}{m}$.

Award ***M0*** for stating sum of roots is $\pm\frac{36}{m}$.

$3\mathrm{i} - 3\mathrm{i} + r = \frac{36}{m}\left(\Rightarrow r = \frac{36}{m}\right)$

substitute their r into product of roots ***(M1)***

$(3\mathrm{i})(-3\mathrm{i})\left(\frac{36}{m}\right) = 4m$ OR $\left(z^2+9\right)\left(\frac{36}{m} - z\right)$

$9 \times \frac{36}{m} = 4m$ OR $\frac{4m}{9} = \frac{36}{m}$

OR

attempt to equate product of three roots to $4m$ ***(M1)***

Note: Accept product of three roots set to $-4m$.

Award ***M0*** for stating product of roots is $\pm 4m$.

$(3\mathrm{i})(-3\mathrm{i}) \times r = 4m\left(\Rightarrow r = \frac{4m}{9}\right)$

substitute their r into sum of roots ***(M1)***

$3\mathrm{i} - 3\mathrm{i} + \frac{4m}{9} = \frac{36}{m}$ OR $\left(z^2+9\right)\left(\frac{4m}{9} - z\right)$

$\frac{4m}{9} = \frac{36}{m}$

THEN

$m = 9$ ***(A1)***

third root is 4 ***A1***

METHOD 2

3i (is a root) ***A1***

(other complex root is) -3i ***A1***

recognition that the other factor is $(z+3\text{i})$ and attempt to write $P(z)$ as product of three linear factors or as product of a quadratic and a linear factor ***(M1)***

$P(z)=(z-3\text{i})(z+3\text{i})(r-z)$ OR
$\left(z-3\text{i}\right)\left(z+3\text{i}\right)=z^2+9\Rightarrow P\left(z\right)=\left(z^2+9\right)\left(\frac{4m}{9}-z\right)$

Note: Accept any attempt at long division of $P(z)$ by z^2+9.

Award ***M0*** for stating other factor is $(z+3\text{i})$ or obtaining z^2+9 with no further working.

Attempt to compare their coefficients ***(M1)***

$-9=-m$ OR $\frac{4m}{9}=\frac{36}{m}$

$m=9$ ***(A1)***

third root is 4 ***A1***

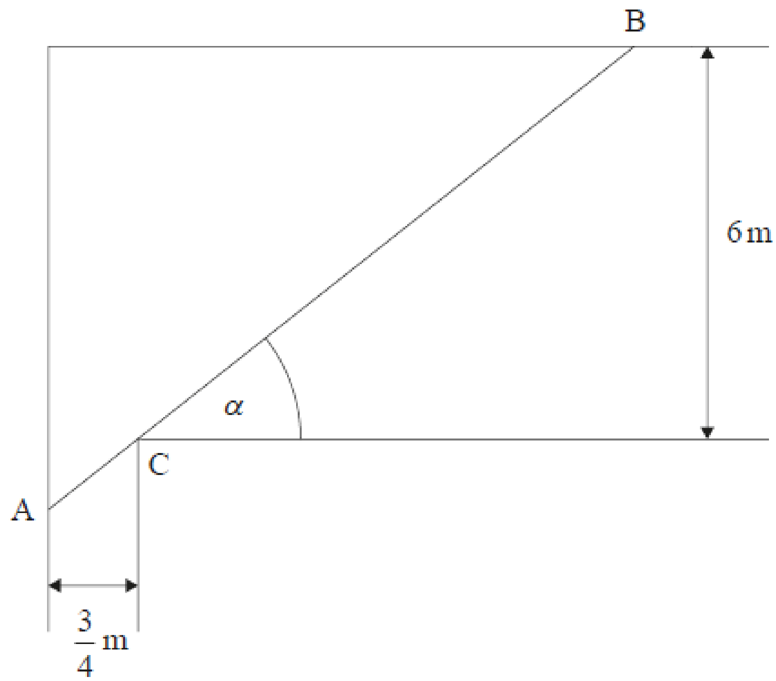
Note: Award a maximum of ***A0A0(M1)(M1)(A1)A1*** for a final answer $P(z)=(z-3\text{i})(z+3\text{i})(4-z)$ seen or stating all three correct factors with no evidence of roots throughout their working.

[6 marks]

29. [Maximum mark: 19] 23M.1.AHL.TZ1.11

Consider the following diagram, which shows the plan of part of a house.

diagram not to scale



A narrow passageway with $\frac{3}{4}$ m width is perpendicular to a room of width 6 m. There is a corner at point C. Points A and B are variable points on the base of the walls such that A, C and B lie on a straight line.

Let L denote the length AB in metres.

Let α be the angle that [AB] makes with the room wall, where $0 < \alpha < \frac{\pi}{2}$.

(a) Show that $L = \frac{3}{4}\sec\alpha + 6\operatorname{cosec}\alpha$. [2]

Markscheme

$L = \text{AC} + \text{CB}$

$\frac{\left(\frac{3}{4}\right)}{\text{AC}} = \cos\alpha \left(\Rightarrow \text{AC} = \frac{\frac{3}{4}}{\cos\alpha} \Rightarrow \text{AC} = \frac{3}{4}\sec\alpha\right)$ ***A1***

$\frac{6}{\text{CB}} = \sin\alpha \left(\Rightarrow \text{CB} = \frac{6}{\sin\alpha} \Rightarrow \text{CB} = 6\operatorname{cosec}\alpha\right)$ ***A1***

so $L = \frac{3}{4}\sec\alpha + 6\operatorname{cosec}\alpha$ ***AG***

[2 marks]

(b.i) Find $\frac{\mathrm{d}L}{\mathrm{d}\alpha}$.

[1]

Markscheme

$\frac{dL}{d\alpha} = \frac{3}{4}\sec\alpha\tan\alpha - 6\operatorname{cosec}\alpha\cot\alpha$ ***A1***

[1 mark]

(b.ii) When $\frac{dL}{d\alpha} = 0$, show that $\alpha = \arctan 2$. [4]

Markscheme

attempt to write $\frac{dL}{d\alpha}$ in terms of $\sin\alpha$, $\cos\alpha$ or $\tan\alpha$ (may be seen in (i)) ***(M1)***

$\frac{dL}{d\alpha} = \frac{\frac{3}{4}\sin\alpha}{\cos^2\alpha} - \frac{6\cos\alpha}{\sin^2\alpha}$ OR $\frac{dL}{d\alpha} = \frac{\frac{3}{4}\tan\alpha}{\cos\alpha} - \frac{6}{\sin\alpha\cos\alpha}\left(= \frac{\frac{3}{4}\tan^3\alpha - 6}{\cos\alpha\tan^2\alpha}\right)$

$\frac{dL}{d\alpha} = 0 \Rightarrow \frac{3}{4}\sin^3\alpha - 6\cos^3\alpha = 0$ OR $\frac{3}{4}\tan^3\alpha - 6 = 0$ (or equivalent) ***(A1)***

$\tan^3\alpha = 8$ ***A1***

$\tan\alpha = 2$ ***A1***

$\alpha = \arctan 2$ ***AG***

[4 marks]

(c.i) Find $\frac{d^2L}{d\alpha^2}$. [3]

Markscheme

attempt to use product rule (at least once) ***(M1)***

$\frac{d^2L}{d\alpha^2} = \frac{3}{4}\sec\alpha\tan\alpha\tan\alpha + \frac{3}{4}\sec\alpha\sec^2\alpha$

$+6\operatorname{cosec}\alpha\cot\alpha\cot\alpha + 6\operatorname{cosec}\alpha\operatorname{cosec}^2\alpha$ ***A1A1***

Note: Award ***A1*** for $\frac{3}{4}\sec\,\alpha\,\tan\,\alpha\,\tan\,\alpha + \frac{3}{4}\sec\,\alpha\,\sec^2\,\alpha$ and ***A1*** for $+6\,\text{cosec}\,\alpha\,\cot\,\alpha\,\cot\,\alpha + 6\,\text{cosec}\,\alpha\,\text{cosec}^2\,\alpha$.

Allow unsimplified correct answer.

$$\left(\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{3}{4}\sec\,\alpha\,\tan^2\,\alpha + \frac{3}{4}\sec^3\,\alpha + 6\,\text{cosec}\,\alpha\,\cot^2\,\alpha + 6\,\text{cosec}^3\,\alpha\right)$$

[3 marks]

(c.ii) When $\alpha = \arctan\,2$, show that $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} = \frac{45}{4}\sqrt{5}$. [4]

Markscheme

attempt to find a ratio other than $\tan\,\alpha$ using an appropriate trigonometric identity OR a right triangle with at least two side lengths seen ***(M1)***

Note: Award ***M0*** for $\alpha = \arctan\,2$ substituted into their $\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}$ with no further progress.

one correct ratio ***(A1)***

$\sec\,\alpha = \sqrt{5}$ OR $\text{cosec}\,\alpha = \frac{\sqrt{5}}{2}$ OR $\cot\,\alpha = \frac{1}{2}$ OR $\cos\,\alpha = \frac{1}{\sqrt{5}}$ OR $\sin\,\alpha = \frac{2}{\sqrt{5}}$

Note: ***M1A1*** may be seen in part (d).

$\frac{3}{4}\left(\sqrt{5}\right)\left(2^2\right) + \frac{3}{4}\left(\sqrt{5}\right)^3 + 6\left(\frac{\sqrt{5}}{2}\right)\left(\frac{1}{2}\right)^2 + 6\left(\frac{\sqrt{5}}{2}\right)^3$ (or equivalent) ***A2***

$\frac{12\sqrt{5}}{4} + \frac{15\sqrt{5}}{4} + \frac{3\sqrt{5}}{4} + \frac{15\sqrt{5}}{4}$

Note: Award ***A1*** for only two or three correct terms.

Award a maximum of ***(M1)(A1)A1*** on ***FT*** from c(i).

$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2}\frac{45}{4}\sqrt{5}$ ***AG***

[4 marks]

(d.i) Hence, justify that L is a minimum when $\alpha = \arctan 2$. [1]

Markscheme

$\frac{\mathrm{d}^2L}{\mathrm{d}\alpha^2} > 0$ OR concave up (or equivalent) ***R1***

(and $\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 0$, when $\alpha = \arctan 2$, hence L is a minimum)

[1 mark]

(d.ii) Determine this minimum value of L. [2]

Markscheme

$(L_{\min} =)\frac{3}{4}\left(\sqrt{5}\right) + 6\left(\frac{\sqrt{5}}{2}\right)$ ***(A1)***

$= \frac{15\sqrt{5}}{4}$ ***A1***

[3 marks]

Two people need to carry a pole of length $11.25\,\mathrm{m}$ from the passageway into the room. It must be carried horizontally.

(e) Determine whether this is possible, giving a reason for your answer. [2]

Markscheme

$\left(11.25 =\right) \frac{15\sqrt{9}}{4} > \frac{15\sqrt{5}}{4}$ (or equivalent comparative reasoning) ***R1***

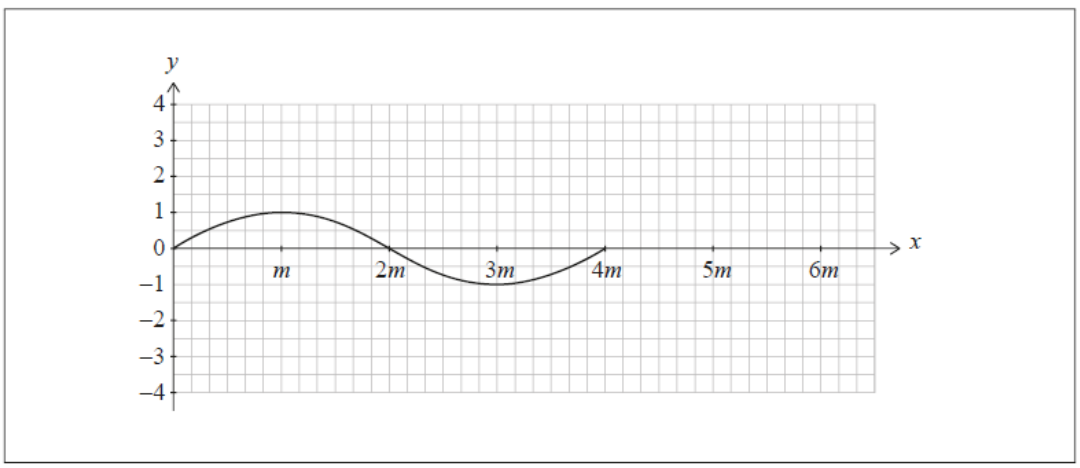
the pole cannot be carried (horizontally from the passageway into the room) ***A1***

Note: Do not award ***R0A1.***

[2 marks]

30. [Maximum mark: 6] 23M.1.AHL.TZ1.5

The function f is defined by $f(x) = \sin qx$, where $q > 0$. The following diagram shows part of the graph of f for $0 \le x \le 4m$, where x is in radians. There are x-intercepts at $x = 0, 2m$ and $4m$.



(a) Find an expression for m in terms of q. [2]

Markscheme

recognition that period is $4m$ OR substitution of a point on f (except the origin) ***(M1)***

$4m = \frac{2\pi}{q}$ OR $1 = \sin qm$

$m = \frac{\pi}{2q}$ ***A1***

[2 marks]

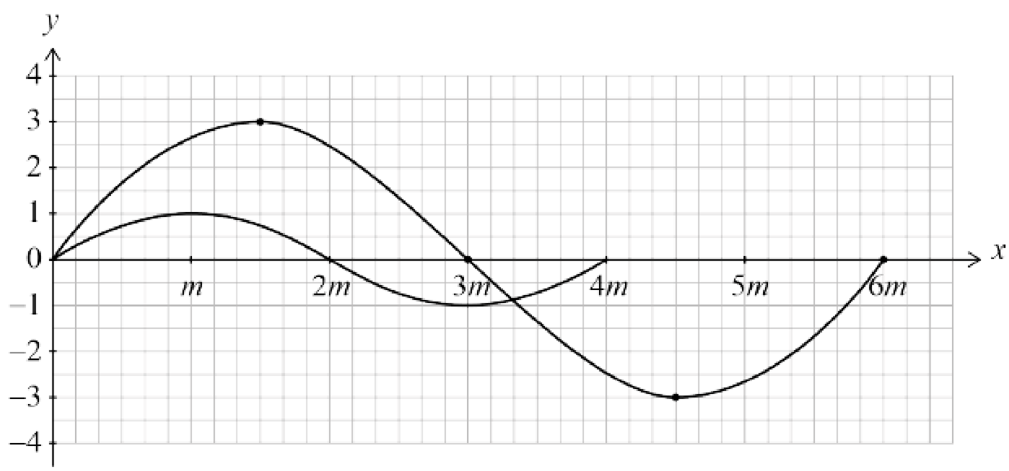
The function g is defined by $g\left(x\right) = 3\sin\frac{2qx}{3}$, for $0 \le x \le 6m$.

(b) On the axes above, sketch the graph of g. [4]

Markscheme

horizontal scale factor is $\frac{3}{2}$ (seen anywhere) ***(A1)***

Note: This ***(A1)*** may be earned by seeing a period of $6m$, half period of $3m$ or the correct x-coordinate of the maximum/minimum point.



A1A1A1

Note:
Curve must be an approximate sinusoidal shape (sine or cosine).
Only in this case, award the following:
A1 for correct amplitude.
A1 for correct domain.
A1 for correct max and min points **and** correct x-intercepts.

[4 marks]

31. [Maximum mark: 14] 23M.1.AHL.TZ1.10

Consider the arithmetic sequence $u_1, u_2, u_3, \ldots$.

The sum of the first n terms of this sequence is given by $S_n = n^2 + 4n$.

(a.i) Find the sum of the first five terms. [2]

Markscheme

recognition that $n = 5$ ***(M1)***

$S_5 = 45$ ***A1***

[2 marks]

(a.ii) Given that $S_6 = 60$, find u_6. [2]

Markscheme

METHOD 1

recognition that $S_5 + u_6 = S_6$ ***(M1)***

$u_6 = 15$ ***A1***

METHOD 2

recognition that $60 = \frac{6}{2}(S_1 + u_6)$ ***(M1)***

$60 = 3(5 + u_6)$

$u_6 = 15$ ***A1***

METHOD 3

substituting their u_1 and d values into $u_1 + (n-1)d$ ***(M1)***

$u_6 = 15$ ***A1***

[2 marks]

(b) Find u_1. [2]

Markscheme

recognition that $u_1 = S_1$ (may be seen in (a)) OR substituting their u_6 into S_6 ***(M1)***

OR equations for S_5 and S_6 in terms of u_1 and d

$1 + 4$ OR $60 = \frac{6}{2}(U_1 + 15)$

$u_1 = 5$ ***A1***

[2 marks]

(c) Hence or otherwise, write an expression for u_n in terms of n. [3]

Markscheme

EITHER

valid attempt to find d (may be seen in (a) or (b)) ***(M1)***

$d = 2$ ***(A1)***

OR

valid attempt to find $S_n - S_{n-1}$ ***(M1)***

$n^2 + 4n - \left(n^2 - 2n + 1 + 4n - 4\right)$ ***(A1)***

OR

equating $n^2 + 4n = \frac{n}{2}\left(5 + u_n\right)$ ***(M1)***

$2n + 8 = 5 + u_n$ (or equivalent) ***(A1)***

THEN

$u_n = 5 + 2(n-1)$ OR $u_n = 2n + 3$ ***A1***

[3 marks]

Consider a geometric sequence, v_n, where $v_2 = u_1$ and $v_4 = u_6$.

(d) Find the possible values of the common ratio, r. [3]

Markscheme

recognition that $v_2 r^2 = v_4$ OR $(v_3)^2 = v_2 \times v_4$ ***(M1)***

$r^2 = 3$ OR $v_3 = \left(\pm\right)5\sqrt{3}$ ***(A1)***

$r = \pm\sqrt{3}$ ***A1***

Note: If no working shown, award ***M1A1A0*** for $\sqrt{3}$.

[3 marks]

(e) Given that $v_{99} < 0$, find v_5. [2]

Markscheme

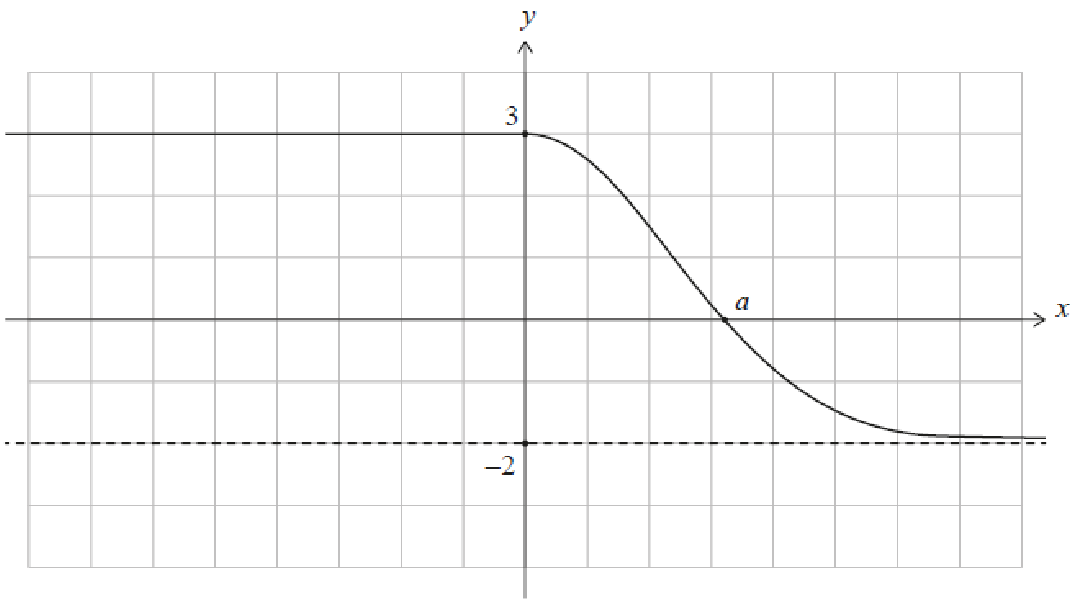
recognition that r is negative ***(M1)***

$v_5 = -15\sqrt{3} \ \left(= -\frac{45}{\sqrt{3}}\right)$ ***A1***

[2 marks]

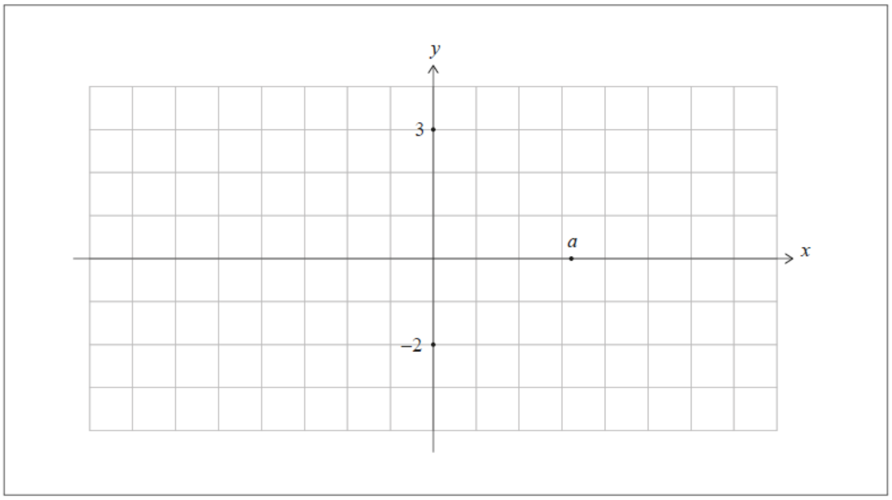
32. [Maximum mark: 7] 23M.1.AHL.TZ1.8

Part of the graph of a function, f, is shown in the following diagram. The graph of $y = f(x)$ has a y-intercept at $(0,\ 3)$, an x-intercept at $(a,\ 0)$ and a horizontal asymptote $y = -2$.



Consider the function $g(x) = |f(|x|)|$.

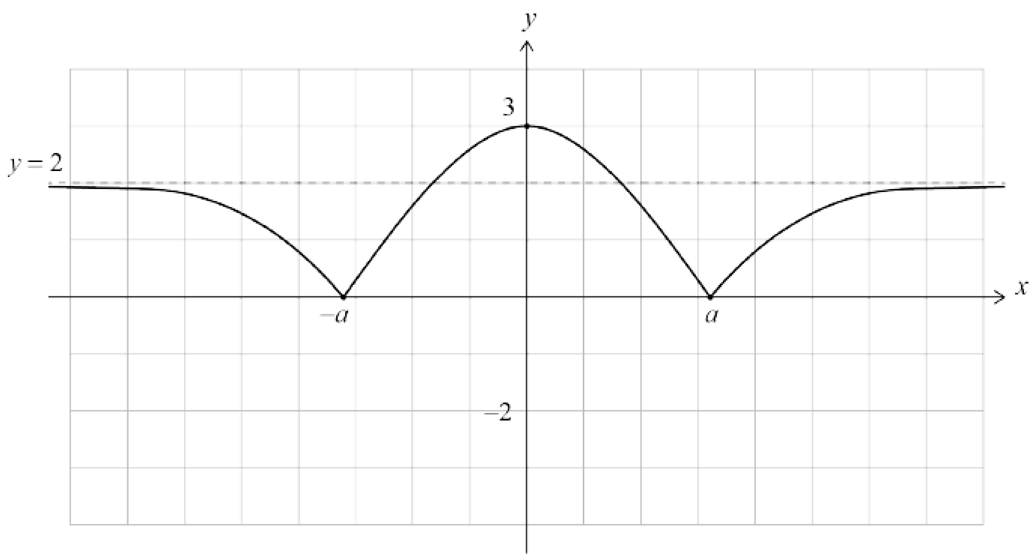
(a) On the following grid, sketch the graph of $y = g(x)$, labelling any axis intercepts and giving the equation of the asymptote.



[4]

Markscheme

attempt to reflect f in the x OR y axis ***(M1)***



A1A1A1

Note: For a curve with an approximately correct shaped right-hand branch, award:

A1 for correct asymptotic behaviour at $y = 2$ (either side)

A1 for correctly reflected RHS of the graph in the y-axis with smooth maximum at $(0,\ 3)$.

A1 for labelled x-intercept at $(-a, 0)$ and labelled asymptote at $y = 2$ with sharp points (cusps) at the x-intercepts.

[4 marks]

(b) Find the possible values of k such that $\left(g(x)\right)^2 = k$ has exactly two solutions.

[3]

Markscheme

$k = 0$ ***A1***

$4 \leq k < 9$ ***A2***

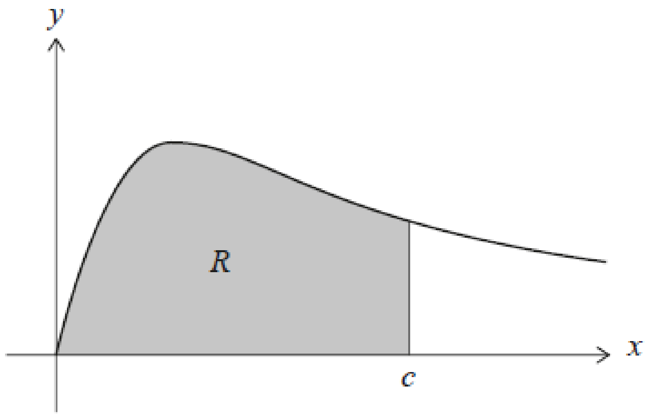
Note: If final answer incorrect, award ***A1*** for critical values 4 and 9 seen anywhere.

Exception to FT:
Award a maximum of ***A0A2FT*** if their graph from (a) is not symmetric about the y-axis.

[3 marks]

33. [Maximum mark: 6] 23M.1.AHL.TZ2.4

The following diagram shows part of the graph of $y = \frac{x}{x^2+2}$ for $x \geq 0$.



[6]

The shaded region R is bounded by the curve, the x-axis and the line $x = c$.

The area of R is $\ln 3$.

Find the value of c.

Markscheme

$A = \int_0^c \frac{x}{x^2+2} \mathrm{d}x$

EITHER

attempts to integrate by inspection or substitution using $u = x^2 + 2$ or $u = x^2$ ***(M1)***

Note: If candidate simply states $u = x^2 + 2$ or $u = x^2$, but does not attempt to integrate, do not award the ***(M1)***.

Note: If candidate does not explicitly state the u-substitution, award the ***(M1)*** only for expressions of the form $k \ln u$ or $k \ln(u+2)$.

$\left[\frac{1}{2}\ln u\right]_2^{c^2+2}$ OR $\left[\frac{1}{2}\ln\left(u+2\right)\right]_0^{c^2}$ OR $\left[\frac{1}{2}\ln\left(x^2+2\right)\right]_0^{c}$ ***A1***

Note: Limits may be seen in the substitution step.

OR

attempts to integrate by inspection ***(M1)***

Note: Award the ***(M1)*** only for expressions of the form $k\ \ln\ \left(x^2+2\right)$.

$\left[\frac{1}{2}\ln\ \left(x^2+2\right)\right]_0^c$ ***A1***

Note: Limits may be seen in the substitution step.

THEN

correctly substitutes their limits into their integrated expression ***(M1)***

$\frac{1}{2}\left(\ln\ \left(c^2+2\right)-\ln\ 2\right)\ \left(=\ln\ 3\right)$ OR $\frac{1}{2}\ln\ \left(c^2+2\right)-\frac{1}{2}\ln\ 2\ \left(=\ln\ 3\right)$

correctly applies at least one log law to their expression ***(M1)***

$\frac{1}{2}\ln\left(\frac{c^2+2}{2}\right)\ \left(=\ln\ 3\right)$ OR $\ln\sqrt{c^2+2}-\ln\sqrt{2}\ \left(=\ln\ 3\right)$ OR $\ln\left(\frac{c^2+2}{2}\right)=\ln\ 9$

OR $\ln\ \left(c^2+2\right)-\ln\ 2-\ln\ 9$ OR $\ln\sqrt{\frac{c^2+2}{2}}\left(=\ln\ 3\right)$ OR $\ln\sqrt{\frac{c^2+2}{\sqrt{2}}}\left(=\ln\ 3\right)$

Note: Condone the absence of $\ln\ 3$ up to this stage.

$\frac{c^2+2}{2}=9$ **OR** $\sqrt{\frac{c^2+2}{2}}=3$ ***A1***

$c^2=16$

$c=4$ ***A1***

Note: Award ***A0*** for $c = \pm 4$ as a final answer.

[6 marks]

34. [Maximum mark: 7] 23M.1.AHL.TZ2.8

The functions f and g are defined by

$f(x) = \cos x, 0 \leq x \leq \frac{\pi}{2}$

$g(x) = \tan x, 0 \leq x < \frac{\pi}{2}$.

The curves $y = f(x)$ and $y = g(x)$ intersect at a point P whose x-coordinate is k, where $0 < k < \frac{\pi}{2}$.

(a) Show that $\cos^2 k = \sin k$. [1]

Markscheme

$\cos k = \frac{\sin k}{\cos k}$ ***A1***

$\cos^2 k = \sin k$ ***AG***

[1 mark]

(b) Hence, show that the tangent to the curve $y = f(x)$ at P and the tangent to the curve $y = g(x)$ at P intersect at right angles. [3]

Markscheme

$f\prime(k) = -\sin k$ and $g\prime(k) = \sec^2 k$ ***A1***

Note: Award ***A1*** for $f\prime(x) = -\sin x$ and $g\prime(x) = \sec^2 x$.

EITHER

$f\prime(k)g\prime(k) = -\frac{\sin k}{\cos^2 k}$ ***M1***

$\cos^2 k = \sin k \Rightarrow f\prime(k)g\prime(k) \left(= -\frac{\sin k}{\sin k}\right) = -1$ ***R1***

OR

$g\prime(k) = \frac{1}{\cos^2 k}$ ***M1***

$\cos^2 k = \sin k \Rightarrow g\prime\left(k\right) = \frac{1}{\sin k} = -\frac{1}{f\prime(k)}$ ***R1***

Note: Accept showing that $f\prime\left(k\right) = -\frac{1}{g\prime(k)}$.

Note: Allow 'backwards methods' such as starting with $f\prime\left(k\right) = -\frac{1}{g\prime(k)}$ leading to $\cos^2 k = \sin k$.

THEN

$\Rightarrow$ the two tangents intersect at right angles at P ***AG***

Note: To obtain the final ***R1***, all of the previous marks must have been awarded.

[3 marks]

(c) Find the value of $\sin k$. Give your answer in the form $\frac{a+\sqrt{b}}{c}$, where a, $c \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. [3]

Markscheme

$1 - \sin^2 k = \sin k$ (from part (a)) ***A1***

$\sin^2 k + \sin k - 1 = 0$

attempts to solve for $\sin\,k$ ***(M1)***

$\sin\,k = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$

(for $0 < k < \frac{\pi}{2}, \sin\,k > 0$) $\Rightarrow \sin\,k = \frac{-1+\sqrt{5}}{2}$ ***A1***

($a = -1, b = 5, c = 2$)

Note: Award ***A0*** if more than one solution is given

[3 marks]