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Mathematics: applications and interpretation

Practice HL paper 3 questions

For first examinations in 2021

Fish

[26 marks]

An estate manager is responsible for stocking a small lake with fish. He begins by introducing 1000 fish into the lake and monitors their population growth to determine the likely carrying capacity of the lake.

After one year an accurate assessment of the number of fish in the lake is taken and it is found to be 1200.

Let N be the number of fish *t* years after the fish have been introduced to the lake.

Initially it is assumed that the rate of increase of *N* will be constant.

a. Use this model to predict the number of fish in the lake when $t = 8$. [2]

When $t = 8$ the estate manager again decides to estimate the number of fish in the lake. To do this he first catches 300 fish and marks them, so they can be recognized if caught again. These fish are then released back into the lake. A few days later he catches another 300 fish, releasing each fish after it has been checked, and finds 45 of them are marked.

b. Assuming the proportion of marked fish in the second sample is equal to the proportion of marked fish in the lake, show that the estate manager will estimate there are now 2000 fish in the lake. [2]

c. Let *X* be the number of marked fish caught in the second sample, where *X* is considered to be distributed as $\mathrm{B}(n, p)$. Assume the number of fish in the lake is 2000.

(i) Write down the value of *n* and the value of *p.*

(ii) State an assumption that is being made for *X* to be considered as following a binomial distribution. [3]

The estate manager decides that he needs bounds for the total number of fish in the lake.

d. (i) Show that an estimate for Var(*X*) is 38.25.

(ii) Hence show that the variance of the proportion of marked fish in the sample, $\mathrm{Var}\left(\frac{X}{300}\right)$, is 0.000425. [4]

The estate manager feels confident that the proportion of marked fish in the lake will be within 1.5 standard deviations of the proportion of marked fish in the sample and decides these will form the upper and lower bounds of his estimate.

e. (i) Taking the value for the variance given in (d) (ii) as a good approximation for the true variance, find the upper and lower bounds for the proportion of marked fish in the lake.

(ii) Hence find upper and lower bounds for the number of fish in the lake when $t = 8$. [4]

f. Given this result, comment on the validity of the linear model used in part (a). [2]

The estate manager now believes the population of fish will follow the logistic model $N(t)=\frac{L}{1+Ce^{-kt}}$ where *L* is the carrying capacity and $C, k > 0$.

The estate manager would like to know if the population of fish in the lake will eventually reach 5000.

g. (i) Assuming a carrying capacity of 5000 use the given values of $N(0)$ and $N(1)$ to calculate the parameters *C* and *k*.

(ii) Use these parameters to calculate the value of $N(8)$ predicted by this model. [7]

h. Comment on the likelihood of the fish population reaching 5000. [2]

Fish markscheme:

(a) $N(8) = 1000 + 200 \times 8$ *M1*

$=2600$ *A1*

[2 marks]

(b) $\frac{45}{300} = \frac{300}{N}$ *M1A1*

$N = 2000$ *AG*

[2 marks]

(c) (i) $n = 300,\ p = \frac{300}{2000} = 0.15$ *A1A1*

(ii) Any valid reason for example: *R1*

Marked fish are randomly distributed, so p constant.

Each fish caught is independent of previous fish caught

[3 marks]

(d) (i) $\text{Var}(X) = np(1-p)$ *M1*

$= 300 \times \frac{300}{2000} \times \frac{1700}{2000}$ *A1*

$= 38.25$ *AG*

(ii) $\text{Var}\left(\frac{X}{300}\right) = \frac{\text{Var}(X)}{300^2}$ *M1A1*

$=0.000425$ *AG*

[4 marks]

(e) (i) $0.15 \pm 1.5\sqrt{0.000425}$ *(M1)*

0.181 and 0.119 *A1*

(ii) $\frac{300}{N} = 0.181\ldots,\quad \frac{300}{N} = 0.119\ldots$ *M1*

Lower bound 1658 upper bound 2519 *A1*

[4 marks]

(f) Linear model prediction falls outside this range so unlikely to be a good model

R1 A1

[2 marks]

(g) (i) $1000 = \frac{5000}{1+C}$ ***M1***

$C = 4$ ***A1***

$1200 = \frac{5000}{1+4e^{-k}}$ ***M1***

$e^{-k} = \frac{3800}{4 \times 1200}$ ***(M1)***

$k = -\ln(0.7916...) = 0.2336...$ ***A1***

(ii) $N(8) = \frac{5000}{1+4e^{-0.2336 \times 8}} = 3090$ ***M1A1***

Note: Accept any answer that rounds to 3000.

[7 marks]

(h) This is much higher than the calculated upper bound for $N(8)$ so the rate of growth of the fish is unlikely to be sufficient to reach a carrying capacity of 5000.

M1R1

[2 marks]

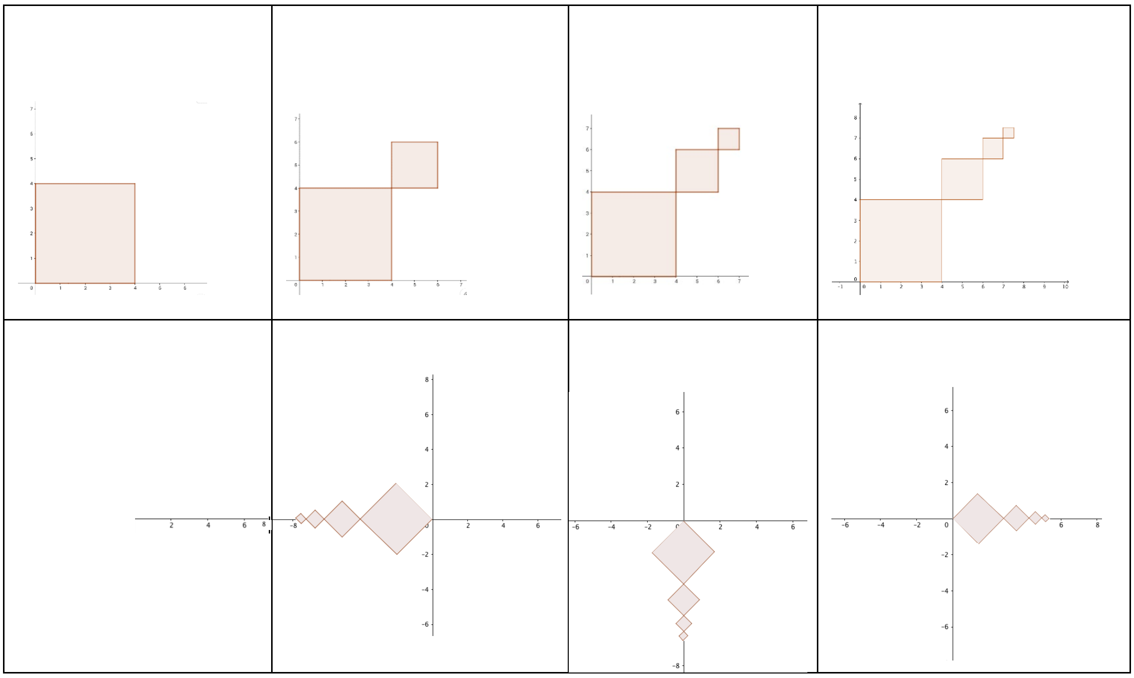
Total [26 marks]

Graphic design

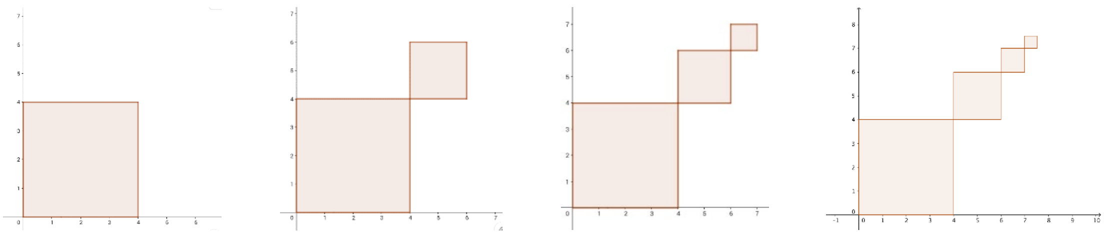
[29 marks]

A graphic designer, Ben, wants to create an animation in which a sequence of squares is created by a composition of successive enlargements and translations and then rotated about the origin and reduced in size.

Ben outlines his plan with the following storyboards.



The first four frames of the animation are shown below in greater detail.



The sides of each successive square are one half the size of the adjacent larger square. Let the sequence of squares be $U_0, U_1, U_2, \ldots$

The first square, U_0, has sides of length 4 cm.

(a) Find an expression for the width of U_n in centimetres. [2]

Ben decides the animation will continue as long as the width of the square is greater than the width of one pixel.

(b) Given the width of a pixel is approximately 0.025 cm, find the number of squares in the final image. [3]

Ben decides to generate the squares using the transformation

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \boldsymbol{A}_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \boldsymbol{b}_n$$

where $\boldsymbol{A}_n$ is a 2×2 matrix that represents an enlargement, $\boldsymbol{b}_n$ is a 2×1 column vector that represents a translation, (x_0, y_0) is a point in U_0 and (x_n, y_n) is its image in U_n.

(c) (i) Write down $\boldsymbol{A}_1$.

(ii) Write down $\boldsymbol{A}_n$, in terms of *n*. [2]

(d) By considering the case where (x_0, y_0) is $(0, 0)$,

(i) state the coordinates, (x_1, y_1), of its image in U_1;

(ii) hence find $\boldsymbol{b}_1$;

(iii) show that $\boldsymbol{b}_n = \begin{pmatrix} 8\left(1-2^{-n}\right) \\ 8\left(1-2^{-n}\right) \end{pmatrix}$. [6]

(e) Hence or otherwise, find the coordinates of the top left-hand corner in U_7. [3]

Once the image of squares has been produced, Ben wants to continue the animation by rotating the image counter clockwise about the origin and having it reduce in size during the rotation.

Let E_θ be the enlargement matrix used when the original sequence of squares has been rotated through θ degrees.

Ben decides the enlargement scale factor, *s,* should be a linear function of the angle, θ, and after a rotation of 360° the sequence of squares should be half of its original length.

(f) (i) Find, *s*, in the form $s(\theta) = m\theta + c$.

(ii) Write down E_θ

(iii) Hence find the image of (1, 1) after it is rotated 135° and enlarged. [9]

(g) Find the value of θ at which the enlargement scale factor equals zero. [1]

After the enlargement scale factor equals zero, Ben continues to rotate the image for another two revolutions.

(h) Describe the animation for these two revolutions, stating the final position of the sequence of squares.

[3]

Graphic design markscheme:

(a) $4\left(\frac{1}{2^n}\right)$ ***M1A1***

[2 marks]

(b) $\frac{4}{2^n} > 0.025$ ***(A1)***

$2^n < 160$

$n \le 7$ ***(A1)***

Note: Accept equations in place of inequalities.

Hence there are 8 squares ***A1***

[3 marks]

(c) (i) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ ***A1***

(ii) $A_n = \begin{pmatrix} \frac{1}{2^n} & 0 \\ 0 & \frac{1}{2^n} \end{pmatrix}$ ***A1***

[2 marks]

(d) (i) (4, 4)
A1

(ii) $\boldsymbol{A}_1\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \boldsymbol{b}_1 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ ***(M1)***

$\boldsymbol{b}_1 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ ***A1***

(iii) Recognise the geometric series $\boldsymbol{b}_n = \begin{pmatrix} 4+2+1+... \\ 4+2+1+... \end{pmatrix}$ ***M1***

Each component is equal to $\dfrac{4\left(1-\frac{1}{2^n}\right)}{\frac{1}{2}}$ $\left(=8\left(1-\frac{1}{2^n}\right)\right)$ ***M1A1***

$$\begin{pmatrix} 8\left(1-\frac{1}{2^n}\right) \\ 8\left(1-\frac{1}{2^n}\right) \end{pmatrix}$$ ***AG***

[6 marks]

(e) $$\begin{pmatrix} \frac{1}{2^7} & 0 \\ 0 & \frac{1}{2^7} \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix}+\begin{pmatrix} 8\left(1-\frac{1}{2^7}\right) \\ 8\left(1-\frac{1}{2^7}\right) \end{pmatrix}$$ ***M1A1***

$(7.9375, 7.96875)$ ***A1***

[3 marks]

(f) (i) $s(\theta)=m\theta+c$

$s(0)=1$, $c=1$ ***M1A1***

$$s(360)=\frac{1}{2}$$ ***A1***

$$\frac{1}{2}=360m+1 \Rightarrow m=-\frac{1}{720}$$ ***A1***

$$s(\theta)=-\frac{\theta}{720}+1$$

(ii) $$E_\theta=\begin{pmatrix} -\frac{\theta}{720}+1 & 0 \\ 0 & -\frac{\theta}{720}+1 \end{pmatrix}$$ ***A1***

(iii) $$\begin{pmatrix} -\frac{135}{720}+1 & 0 \\ 0 & -\frac{135}{720}+1 \end{pmatrix}\begin{pmatrix} \cos 135^\circ & -\sin 135^\circ \\ \sin 135^\circ & \cos 135^\circ \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ ***M1A1A1***

(-1.15, 0) ***A1***

[9 marks]

(g) $\theta=720^\circ$ ***A1***

[1 mark]

(h) The image will expand from zero (accept equivalent answers)

It will rotate counter clockwise

The design will (re)appear in the opposite (third) quadrant ***A1A1***

Note: Accept any two of the above

Its final position will be in the opposite (third) quadrant or 180˚ from its original position or equivalent statement. ***A1***

[3 marks]

Total [29 marks]