Algorithms - revision 1 [112 marks]

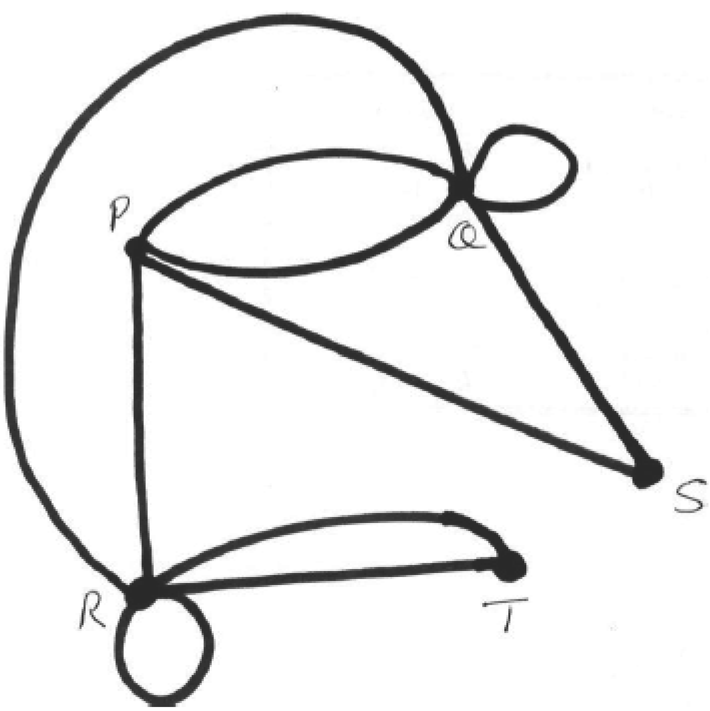
1. [Maximum mark: 19] EXM.2.AHL.TZ0.18

The adjacency matrix of the graph G, with vertices P, Q, R, S, T is given by:

$$\begin{array}{c} \\ \text{P} \\ \text{Q} \\ \text{R} \\ \text{S} \\ \text{T} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{P} & \text{Q} & \text{R} & \text{S} & \text{T} \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix} \end{array}$$

(a) Draw the graph of G. [3]

Markscheme



A3

Note: Award A2 for one missing or misplaced edge,

A1 for two missing or misplaced edges.

[3 marks]

(b.i) Define an Eulerian circuit. [1]

Markscheme

an Eulerian circuit is one that contains every edge of the graph exactly once ***A1***

[1 mark]

(b.ii) Write down an Eulerian circuit in G starting at P. [2]

Markscheme

a possible Eulerian circuit is

P → Q → S → P → Q → Q → R → T → R → R → P ***A2***

[2 marks]

(c.i) Define a Hamiltonian cycle. [2]

Markscheme

a Hamiltonian cycle passes through each vertex of the graph ***A1***

exactly once ***A1***

[2 marks]

(c.ii) Explain why it is not possible to have a Hamiltonian cycle in G. [3]

Markscheme

to pass through T, you must have come from R and must return to R. ***R3***

hence there is no Hamiltonian cycle

[3 marks]

(d.i) Find the number of walks of length 5 from P to Q. [4]

Markscheme

using the adjacency matrix $\boldsymbol{A} = \begin{pmatrix} 0 & 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}$, ***(M1)***

we need the entry in the first row second column of the matrix $\boldsymbol{A}^5$ ***(M1)***

$$\boldsymbol{A}^5 = \begin{pmatrix} 245 & 309 & 274 & 143 & 126 \\ 309 & 363 & 322 & 168 & 156 \\ 274 & 322 & 295 & 141 & 164 \\ 143 & 168 & 141 & 77 & 72 \\ 126 & 156 & 164 & 72 & 72 \end{pmatrix}$$ ***(A1)***

hence there are 309 ways ***A1***

[4 marks]

(d.ii) Which pairs of distinct vertices have more than 15 walks of length 3 between them? [4]

Markscheme

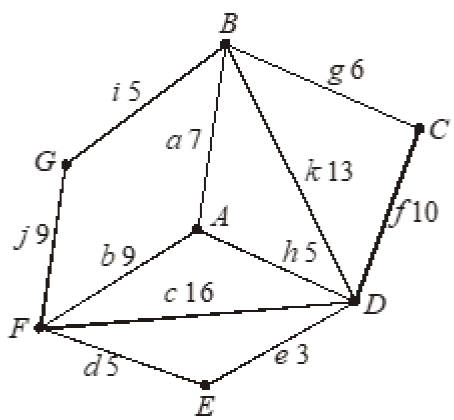
$$A^3 = \begin{pmatrix} 13 & 21 & 17 & 10 & 6 \\ 21 & 22 & 19 & 11 & 8 \\ 17 & 19 & 18 & 7 & 14 \\ 10 & 11 & 7 & 5 & 4 \\ 6 & 8 & 14 & 4 & 4 \end{pmatrix} \quad \textit{(M1)}$$

hence the pairs of vertices are PQ, PR and QR ***A1A1A1***

[4 marks]

2. [Maximum mark: 8] EXM.1.AHL.TZ0.36

Apply Prim's algorithm to the weighted graph given below to obtain the minimal spanning tree starting with the vertex A.



Find the weight of the minimal spanning tree. [8]

Markscheme

We start with point A and write S as the set of vertices and T as the set of edges.
The weights on each edge will be used in applying Prim's algorithm.
Initially, $S = \{A\}$, $T = \emptyset$. In each subsequent stage, we shall update S and T.

Step 1: Add edge h: So $S = \{A, D\}$, $T = \{h\}$

Step 2: Add edge e: So $S = \{A, D, E\}$ $T = \{h, e\}$

Step 3: Add edge d: Then $S = \{A, D, E, F\}$ $T = \{h, e, d\}$

Step 4: Add edge a: Then $S = \{A, D, E, F, B\}$ $T = \{h, e, d, a\}$

Step 5: Add edge i: Then $S = \{A, D, E, F, B, G\}$ $T = \{h, e, d, a, i\}$

Step 6: Add edge g: Then $S = (A, D, E, F, B, G, C\}$ $T = \{h, e, d, a, I, g\}$ ***(M4)(A3)***

Notes: Award ***(M4)(A3)*** for all 6 correct,

(M4)(A2) for 5 correct;

(M3)(A2) for 4 correct,

(M3)(A1) for 3 correct;

(M1)(A1) for 2 correct,

(M1)(A0) for 1 correct

OR

(M2) for the correct definition of Prim's algorithm,

(M2) for the correct application of Prim's algorithm,

(A3) for the correct answers at the last three stages.

Now S has all the vertices and the minimal spanning tree is obtained.

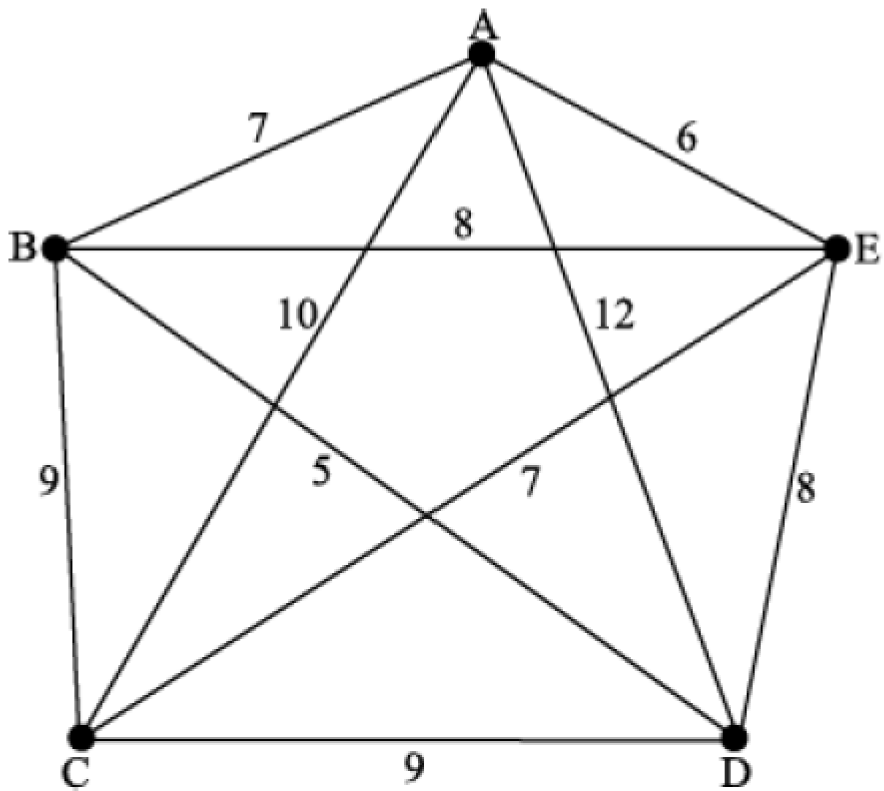
The weight of the edges in T is $5 + 3 + 5 + 7 + 5 + 6$

$= 31$ ***(A1)***

[8 marks]

3. [Maximum mark: 14] EXM.2.AHL.TZ0.19

Let G be the graph below.



(a) Find the total number of Hamiltonian cycles in *G*, starting at vertex A. Explain your answer. [3]

Markscheme

Starting from vertex A there are 4 choices. From the next vertex there are three choices, etc… ***M1R1***

So the number of Hamiltonian cycles is 4! = 24. ***A1 N1***

[3 marks]

(b.i) Find a minimum spanning tree for the subgraph obtained by deleting A from *G*. [3]

Markscheme

Start (for instance) at B, using Prim´s algorithm Then D is the nearest vertex ***M1***

Next E is the nearest vertex ***A1***

Finally C is the nearest vertex So a minimum spanning tree is B → D → E → C ***A1 N1***

[3 marks]

(b.ii) Hence, find a lower bound for the travelling salesman problem for G. [3]

Markscheme

A lower bound for the travelling salesman problem is then obtained by adding the weights of AB and AE to the weight of the minimum ***M1***

spanning tree (ie 20) ***A1***

A lower bound is then $20 + 7 + 6 = 33$ ***A1 N1***

[3 marks]

(c) Give an upper bound for the travelling salesman problem for the graph above. [2]

Markscheme

ABCDE is an Hamiltonian cycle ***A1***

Thus an upper bound is given by $7 + 9 + 9 + 8 + 6 = 39$ ***A1***

[2 marks]

(d) Show that the lower bound you have obtained is not the best possible for the solution to the travelling salesman problem for G. [3]

Markscheme

Eliminating C from G a minimum spanning tree is E → A → B → D ***M1***

of weight 18 ***A1***

Adding BC to CE$(18 + 9 + 7)$ gives a lower bound of $34 > 33$ ***A1***

So 33 not the best lower bound. ***AG N0***

[3 marks]

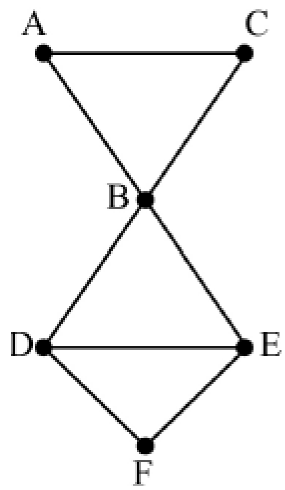
4. [Maximum mark: 17] SPM.2.AHL.TZ0.5

The following table shows the costs in US dollars (US$) of direct flights between six cities. Blank cells indicate no direct flights. The rows represent the departure cities. The columns represent the destination cities.

		Destination city					
		A	**B**	**C**	**D**	**E**	**F**
Departure city	**A**		90	150			
	B	90		80	70	140	
	C	150	80				
	D		70			100	180
	E		140		100		210
	F				180	210	

(a) Show the direct flights between the cities as a graph. [2]

Markscheme



A2

[2 marks]

(b) Write down the adjacency matrix for this graph. [2]

Markscheme

attempt to form an adjacency matrix ***M1***

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$ ***A1***

[2 marks]

(c) Using your answer to part (b), find the number of different ways to travel from and return to city A in exactly 6 flights. [2]

Markscheme

raising the matrix to the power six ***(M1)***

50 ***A1***

[2 marks]

(d) State whether or not it is possible to travel from and return to city A in exactly 6 flights, having visited each of the other 5 cities exactly once. Justify your answer. [2]

Markscheme

not possible ***A1***

because you must pass through B twice ***R1***

Note: Do not award ***A1R0***.

[2 marks]

The following table shows the least cost to travel between the cities.

		Destination city					
		A	**B**	**C**	**D**	**E**	**F**
Departure city	**A**	0	90	150	160	a	b
	B	90	0	80	70	140	250
	C	150	80	0	150	220	330
	D	160	70	150	0	100	180
	E	a	140	220	100	0	210
	F	b	250	330	180	210	0

(e) Find the values of a and b. [2]

Markscheme

$a = 230, b = 340$ ***A1A1***

[2 marks]

A travelling salesman has to visit each of the cities, starting and finishing at city A.

(f) Use the nearest neighbour algorithm to find an upper bound for the cost of the trip. [3]

Markscheme

A → B → D → E → F → C → A ***(M1)***

90 + 70 + 100 + 210 + 330 + 150 ***(A1)***

(US$) 950 ***A1***

[3 marks]

(g) By deleting vertex A, use the deleted vertex algorithm to find a lower bound for the cost of the trip. [4]

Markscheme

finding weight of minimum spanning tree ***M1***

70 + 80 + 100 + 180 = (US$) 430 ***A1***

adding in two edges of minimum weight ***M1***

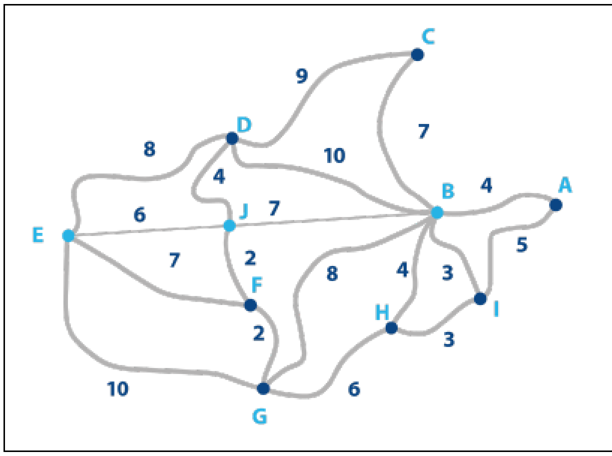
430 + 90 + 150 = (US$) 670 ***A1***

[4 marks]

5. [Maximum mark: 7] EXN.1.AHL.TZ0.11

Nymphenburg Palace in Munich has extensive grounds with 9 points of interest (stations) within them.

These nine points, along with the palace, are shown as the vertices in the graph below. The weights on the edges are the walking times in minutes between each of the stations and the total of all the weights is 105 minutes.



A	Nymphenburg Palace
B	Fountain
C	Magdalenenklause
D	Pagodenburg
E	Large Cascade
F	Pan with Spring
G	Badenburg
H	Amalienburg
I	Summerhouse
J	Bridge

Anders decides he would like to walk along all the paths shown beginning and ending at the Palace (vertex A).

Use the Chinese Postman algorithm, clearly showing all the stages, to find the shortest time to walk along all the paths. [7]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

Odd vertices are B, F, H and I **(M1)A1**

Pairing the vertices **M1**

BF and HI $9+3=12$
BH and FI $4+11=15$
BI and FH $3+8=11$ **A2**

Note: award **A1** for two correct totals.

Shortest time is $105+11=116$ (minutes) **M1A1**

[7 marks]

6. [Maximum mark: 8] EXM.1.AHL.TZ0.37

In this part, marks will only be awarded if you show the correct application of the required algorithms, and show all your working.

In an offshore drilling site for a large oil company, the distances between the planned wells are given below in metres.

	1	2	3	4	5	6	7	8	9	10
2	30									
3	40	60								
4	90	190	130							
5	80	200	10	160						
6	70	40	20	40	130					
7	60	120	50	90	30	60				
8	50	140	90	70	140	70	40			
9	40	170	140	60	50	90	50	70		
10	200	80	150	110	90	30	190	90	100	
11	150	30	200	120	190	120	60	190	150	200

It is intended to construct a network of paths to connect the different wells in a way that minimises the sum of the distances between them.

Use Prim's algorithm, starting at vertex 3, to find a network of paths of minimum total length that can span the whole site. [8]

Markscheme

Vertices added to the Tree	Edge added	Weight
3	∅	0
5	3, 5	10
6	3, 6	20
7	5, 7	30
10	6, 10	30
1	3, 1	40
2	1, 2	30
11	2, 11	30
9	1, 9	40
4	6, 4	40
8	7, 8	40
		310

(R2)(A4)(M1)

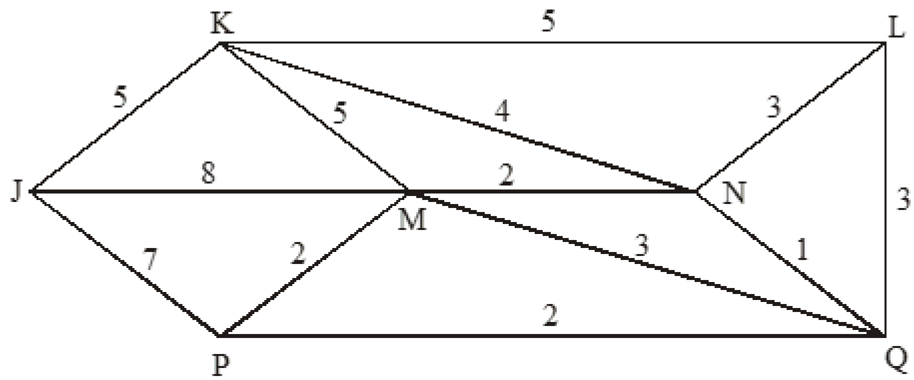
(A1)

Note: Award ***(R2)*** for correct algorithms, ***(R1)*** for 1 error, ***(R0)*** for 2 or more errors.
Award ***(A4)*** for correct calculations, ***(A3)*** for 1 error, ***(A2)*** for 2 errors, ***(A1)*** for 3 errors, ***(A0)*** for 4 or more errors.
Award ***(M1)*** for tree/table/method.
Award ***(A1)*** for minimum weight.

[8 marks]

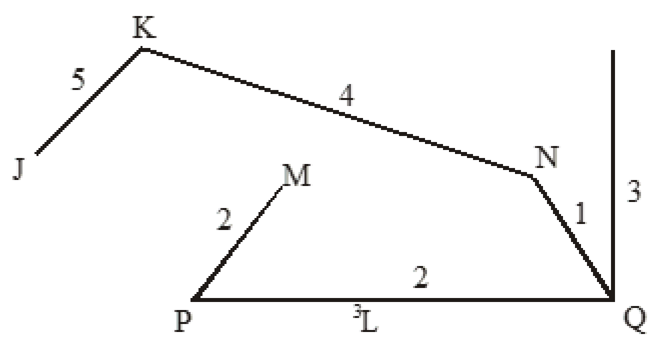
7. [Maximum mark: 6] EXM.1.AHL.TZ0.38

The diagram below shows a weighted graph.



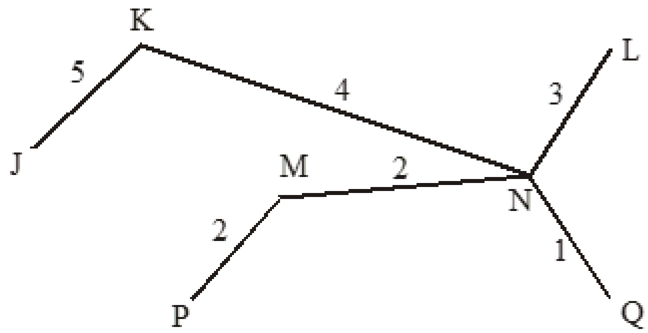
Use Prim's algorithms to find a minimal spanning tree, starting at J.
Draw the tree, and find its total weight. [6]

Markscheme



(C4)

OR



(C4)

Total weight = 17 ***(A2)***

Note: There are other possible spanning trees.

[6 marks]

8. [Maximum mark: 10] EXM.1.AHL.TZ0.40

The weights of the edges of a complete graph G are shown in the following table.

	A	B	C	D	E	F
A	–	5	4	7	6	2
B	5	–	6	3	5	4
C	4	6	–	8	1	6
D	7	3	8	–	7	3
E	6	5	1	7	–	3
F	2	4	6	3	3	–

Starting at B, use Prim's algorithm to find and draw a minimum spanning tree for G. Your solution should indicate the order in which the vertices are added. State the total weight of your tree. [10]

Markscheme

Different notations may be used but the edges should be added in the following order.

Using Prim's Algorithm, ***(M1)***

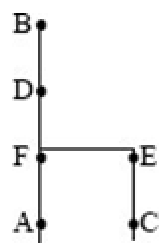
BD ***A1***

DF ***A1***

FA ***A1***

FE ***A1***

EC ***A1***

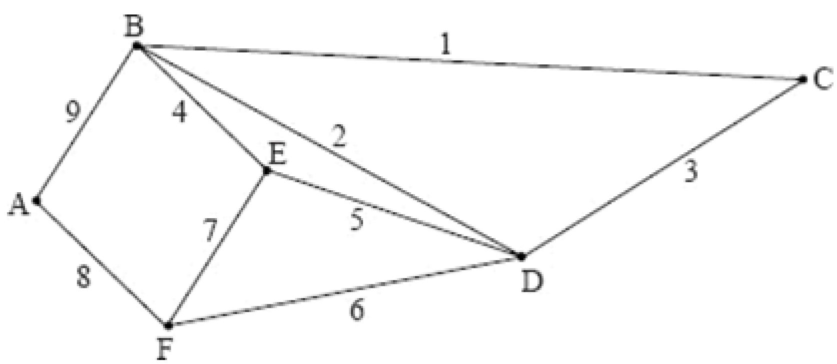


A2

Total weight = 12 ***A2***

[10 marks]

9. [Maximum mark: 9] EXM.1.AHL.TZ0.24



The above diagram shows the weighted graph G.

(a.i) Write down the adjacency matrix for G. [1]

Markscheme

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad \textit{A1}$$

[1 mark]

(a.ii) Find the number of distinct walks of length 4 beginning and ending at A. [3]

Markscheme

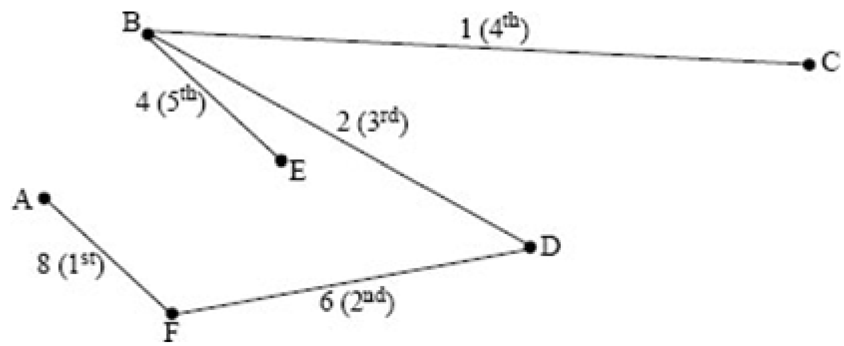
We require the (A, A) element of M^4 which is 13. ***M1A2***

[3 marks]

(b) Starting at A, use Prim's algorithm to find and draw the minimum spanning tree for G.

Your solution should indicate clearly the way in which the tree is constructed. [5]

Markscheme



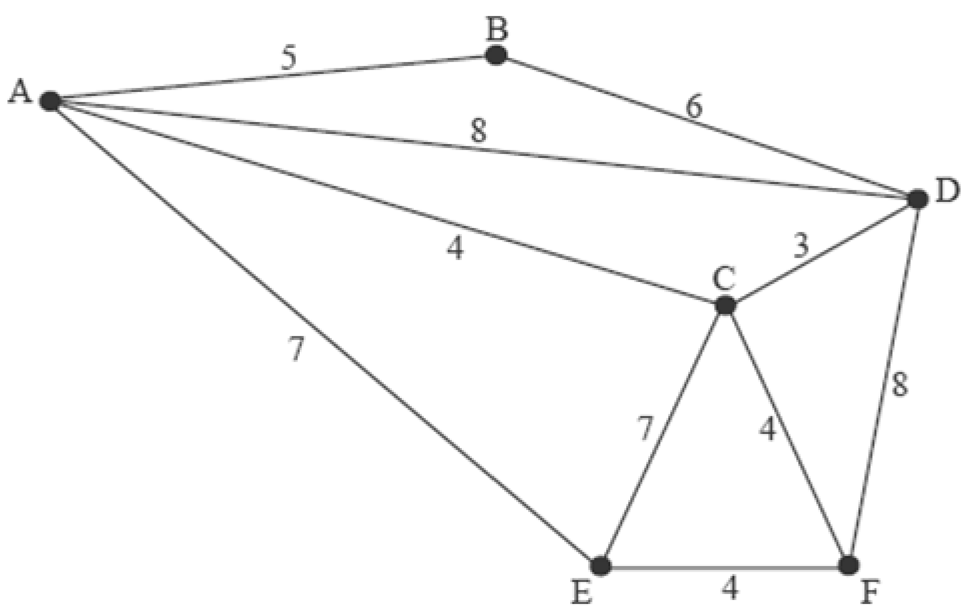
A1A1A1A1A1

[5 marks]

10. [Maximum mark: 8] EXM.1.AHL.TZ0.23

Sameer is trying to design a road system to connect six towns, A, B, C, D, E and F.

The possible roads and the costs of building them are shown in the graph below. Each vertex represents a town, each edge represents a road and the weight of each edge is the cost of building that road. He needs to design the lowest cost road system that will connect the six towns.



(a) Name an algorithm that will allow Sameer to find the lowest cost road system. [1]

Markscheme

EITHER

Prim's algorithm ***A1***

OR

Kruskal's algorithm ***A1***

[1 mark]

(b) Find the lowest cost road system and state the cost of building it. Show clearly the steps of the algorithm. [7]

Markscheme

EITHER

using Prim's algorithm, starting at A

Edge	Cost
AC	4
CD	3
CF	4
FE	4
AB	5

A1A1A1A1A1

lowest cost road system contains roads AC, CD, CF, FE and AB ***A1***

cost is 20 ***A1***

OR

using Kruskal's algorithm

Edge	Cost
CD	3
CF	4
FE	4
AC	4
AB	5

A1A1A1A1A1

lowest cost road system contains roads CD, CF, FE, AC and AB ***A1***

cost is 20 ***A1***

Note: Accept alternative correct solutions.

[7 marks]

11. [Maximum mark: 6] EXM.1.AHL.TZ0.1

The cost adjacency matrix below represents the distance in kilometres, along routes between bus stations.

	A	B	C	D	E
A	-	x	2	6	p
B	x	-	5	7	q
C	2	5	-	3	r
D	6	7	3	-	s
E	p	q	r	s	-

All the values in the matrix are positive, distinct integers.

It is decided to electrify some of the routes, so that it will be possible to travel from any station to any other station solely on electrified routes. In order to achieve this with a minimal total length of electrified routes, Prim's algorithm for a minimal spanning tree is used, starting at vertex A.

The algorithm adds the edges in the following order:

AB AC CD DE.

There is only one minimal spanning tree.

(a) Find with a reason, the value of x. [2]

Markscheme

AB must be the length of the smallest edge from A so $x = 1$. ***R1A1***

[2 marks]

(b) If the total length of the minimal spanning tree is 14, find the value of s. [2]

Markscheme

$1 + 2 + 3 + s = 14 \Rightarrow s = 8$ ***M1A1***

[2 marks]

(c) Hence, state, with a reason, what can be deduced about the values of p, q, r. [2]

Markscheme

The last minimal edge chosen must connect to E , so since $s = 8$ each of p , q, r must be ≥ 9. ***R1A1***

[2 marks]