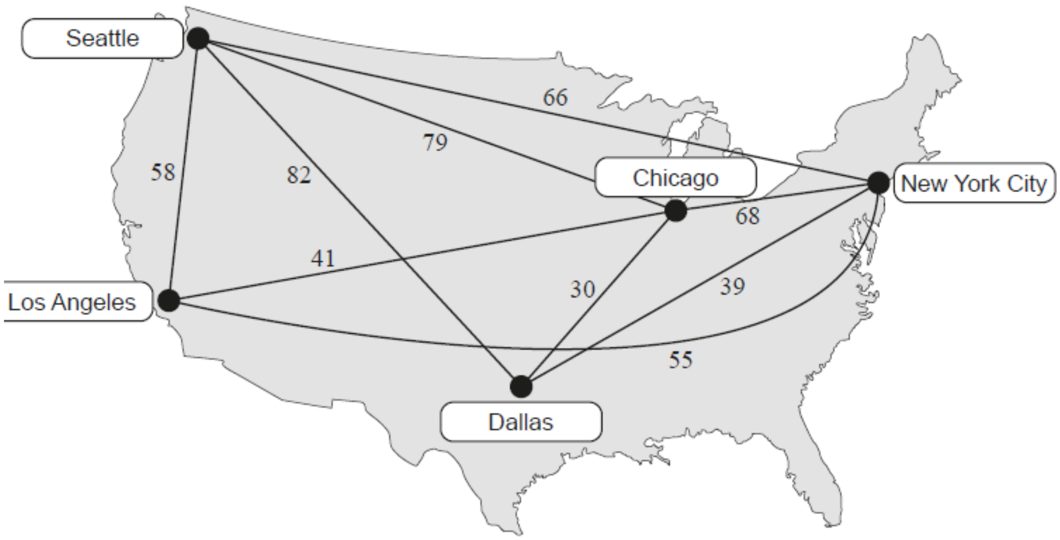
Algorithms - revision 2 [91 marks]

1. [Maximum mark: 19] 23M.2.AHL.TZ2.4

The following graph shows five cities of the USA connected by weighted edges representing the cheapest direct flights in dollars ($\$$) between cities.



(a) Explain why the graph can be described as "connected", but not "complete". [2]

Markscheme

any city can be travelled to or from any other city (so is connected) ***R1***

EITHER

but there is no direct flight between Los Angeles and Dallas (for example) ***R1***

OR

but not every vertex has degree 4 ***R1***

Note: Accept equivalent statements for the cities being connected and the graph not being complete.

[2 marks]

(b) Find a minimum spanning tree for the graph using Kruskal's algorithm.

State clearly the order in which your edges are added, and draw the tree obtained. [3]

Markscheme

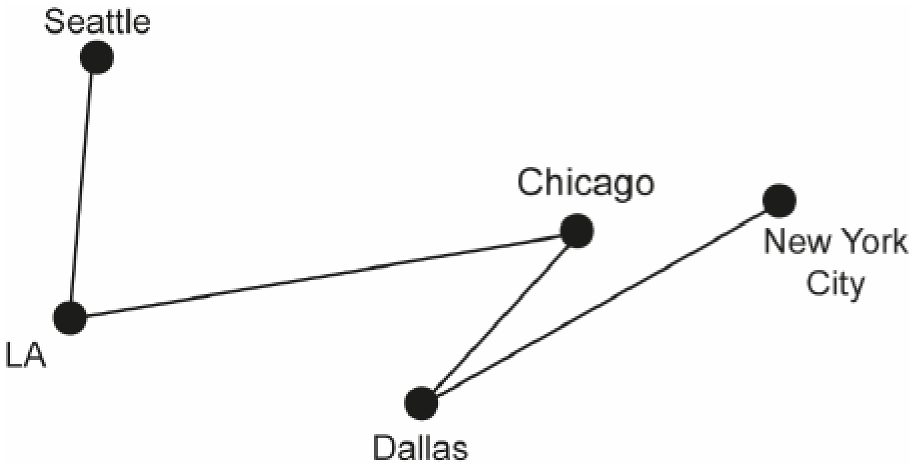
edge CD selected first ***M1***

DN,

CL,

LS ***A1***

Note: Award marks if the answers are written as sums in the correct order. ***M1*** if 30 is seen first, ***A1*** for $30 + 39 + 41 + 58$.



A1

Note: The final ***A1*** can be awarded independently. Award ***M0A0A1*** for a correct MST graph with no other working. Award ***M1A0A1*** if Prim's algorithm is seen to be used correctly with CD first.

[3 marks]

(c) Using only the edges obtained in your answer to part (b), find an upper bound for the travelling salesman problem. [2]

Markscheme

$2 \times \mathrm{MST}$ weight ***(M1)***

$= \$336$ ***A1***

Note: Allow any integer multiple (> 1) of MST weight for ***M1,*** and if correctly calculated, award ***M1A1.***

[2 marks]

Ronald lives in New York City and wishes to fly to each of the other cities, before finally returning to New York City. After some research, he finds that there exists a direct flight between Los Angeles and Dallas costing $\$26$. He updates the graph to show this.

(d) By using the nearest neighbour algorithm and starting at Los Angeles, determine a better upper bound than that found in part (c).

State clearly the order in which you are adding the vertices. [3]

Markscheme

attempt at nearest neighbour algorithm ***M1***

order is $\mathrm{LA} \to \mathrm{D} \to \mathrm{C} \to \mathrm{NYC} \to \mathrm{S} \to \mathrm{LA}$ ***A1***

Note: Award ***M1*** for a route that begins with LA and then D, this includes seeing 26 as the first value in a sum. Award ***A1*** if $26 + 30 + 68 + 66 + 58$ seen in order.

Note: Award ***M1A0*** for an incorrect first nearest neighbour proceeding 'correctly' to the next vertex. For example, LA to C and then C to D.

upper bound is $(26 + 30 + 68 + 66 + 58 =)\ \$248$ ***A1***

Note: Award ***M1A0*** for correct nearest neighbour algorithm starting from a vertex other than LA. Condone the correct tour written backwards i.e. $58 + 66 + 68 + 30 + 26 = 248$

[3 marks]

(e.i) By deleting the vertex which represents Chicago, use the deleted vertex algorithm to determine a lower bound for the travelling salesman problem. [3]

Markscheme

attempt to find MST of L, N, D and S ***(M1)***

by deleting C, Kruskal gives MST for the remainder as LD, DN, LS weight 123 ***(A1)***

(lower bound is therefore $123+(30+41)=)\$194$ ***A1***

Note: Award ***(M1)*** for a graph or list of edges that does not include C.

Award ***(A1)*** if $26+39+58$ seen in any order.

[3 marks]

(e.ii) Similarly, by instead deleting the vertex which represents Seattle, determine another lower bound. [2]

Markscheme

by deleting S, Kruskal gives MST for the remainder as LD, DC, DN weight 95 ***(A1)***

(lower bound is therefore $95+(58+66)=)$ $\$219$ ***A1***

Note: Award ***(A1)*** if $26 + 30 + 39$ seen in any order.

[2 marks]

(f) Hence, using your previous answers, write down your best inequality for the **least** expensive tour Ronald could take. Let the variable C represent the total cost, in dollars, for the tour. [2]

Markscheme

$219 \leq C \leq 248$ ***A1A1***

Note: Award ***A1*** for $219 \leq C$ and ***A1*** for $C \leq 248$. Award at most ***A1A0*** for $219 < C < 248$. ***FT*** for their values from part (e) if higher value from (e)(i) and (e)(ii) used for the lower bound, and part (d) for the upper.

[2 marks]

(g) Write down a tour that is strictly greater than your lower bound and strictly less than your upper bound. [2]

Markscheme

any valid tour, within their interval from part (f), from any starting point **OR** any valid tour that starts and finishes at N ***(M1)***

valid tour starting point N **AND** within their interval ***A1***
e.g. NDCLSN (weight 234)

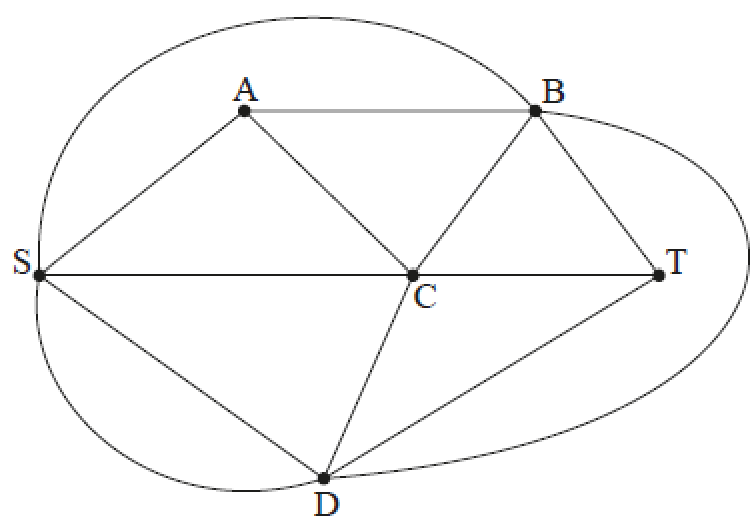
Note: If part (f) not correct, only award ***A1FT*** if their valid tour begins and ends at N **AND** lies within **BOTH** their interval (including if one-sided) in part (f) **AND** $219 \leq C \leq 248$.

If no response in the form of an interval seen in part (f) then award ***M1A0*** for a valid tour beginning and ending at N **AND** within $219 \leq C \leq 248$.

[2 marks]

2. [Maximum mark: 7] 22N.1.AHL.TZ0.4

In a competition, a contestant has to move through a maze to find treasure. A graph of the maze is shown below, where each edge represents a corridor in the maze. The contestant starts at S and the treasure is located at T.



(a) Complete the adjacency matrix, $\boldsymbol{M}$, for the graph.

	S	A	B	C	D	T
S	0	1	1	1	[ ]	0
A	1	0	1	1	[ ]	0
B	1	1	0	1	1	1
C	1	1	1	0	1	1
D	[ ]	[ ]	1	1	0	1
T	0	0	1	1	1	0

[2]

Markscheme

	S	A	B	C	D	T
S	0	1	1	1	**2**	0
A	1	0	1	1	**0**	0
B	1	1	0	1	1	1
C	1	1	1	0	1	1
D	**2**	**0**	1	1	0	1
T	0	0	1	1	1	0

$\mathrm{SD} = \mathrm{DS} = 2$ ***A1***

$\mathrm{AD} = \mathrm{DA} = 0$ ***A1***

[2 marks]

The competition rules state that the contestant can walk along a maximum of four corridors.

(b) Find the number of walks from S to T with a maximum of 4 edges. [4]

Markscheme

attempt to calculate at least one of $\boldsymbol{M}^2$, $\boldsymbol{M}^3$ and $\boldsymbol{M}^4$ ***(M1)***

attempt to calculate all of $\boldsymbol{M}^2$, $\boldsymbol{M}^3$ and $\boldsymbol{M}^4$ ***(M1)***

finding at least one of the top right entries, $4,\ 10,\ 64$ ***(A1)***

78 walks ***A1***

Note: If $\mathrm{SD} = \mathrm{DS} = 1$ is their answer in part (a), their ***FT*** answer is $(3 + 8 + 41 =)\ 52$ walks.

[4 marks]

(c) Explain why the number of ways the contestant can reach the treasure is less than the answer to part (b). [1]

Markscheme

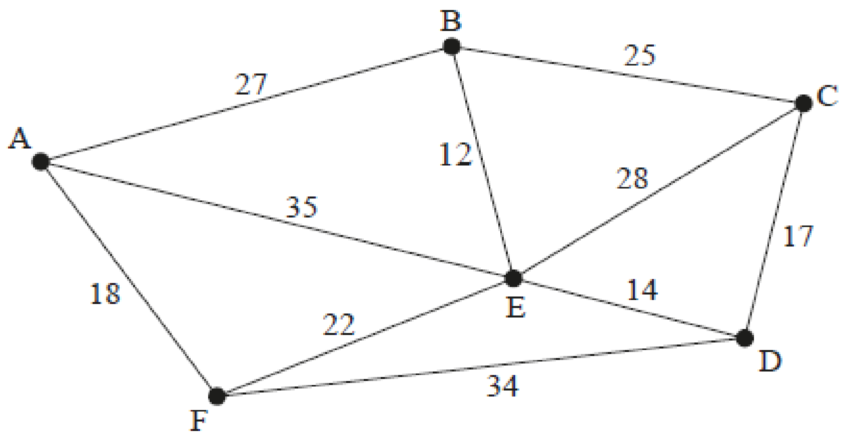
because some of the walks will pass through T, before returning to T ***R1***

[1 mark]

3. [Maximum mark: 14] 22N.2.AHL.TZ0.4

A company has six offices, A, B, C, D, E and F. One of the company managers, Nanako, needs to visit the offices. She creates the following graph that shows the distances, in kilometres, between some of the offices.

diagram not to scale



(a) Write down a Hamiltonian cycle for this graph. [1]

Markscheme
any correct Hamiltonian cycle e.g. $\mathrm{ABCDEFA}$ ***A1***   ***[1 mark]***

(b) State, with a reason, whether the graph contains an Eulerian circuit. [1]

Markscheme
no, since not all vertices have an even degree (or equivalent) ***R1***   ***[1 mark]***

Nanako wishes to find the shortest cycle to visit all the offices. She decides to complete a weighted adjacency table, showing the least distance between each pair of offices.

	A	B	C	D	E	F
A		27	52	p	35	18
B			25	26	12	q
C				17	28	r
D					14	34
E						22
F						

Write down the value of

(c.i) p. [1]

Markscheme

49 ***A1***

[1 mark]

(c.ii) q. [1]

Markscheme

34 ***A1***

[1 mark]

(c.iii) r. [1]

Markscheme

50 *A1*

[1 mark]

(d) Starting at vertex E, use the nearest neighbour algorithm to find an upper bound for Nanako's cycle. [3]

Markscheme

cycle is $\mathrm{EBCDFAE}$ *(M1)(A1)*

$\mathrm{UB} = 12 + 25 + 17 + 34 + 18 + 35$

Note: Award ***M1*** for $12 + 25 + 17 + \ldots$ **OR** EBCD.

$= 141$ *A1*

[3 marks]

(e) By deleting vertex F, find a lower bound for Nanako's cycle. [4]

Markscheme

attempt to find MST for vertices $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ and E *M1*

$12 + 14 + 17 + 27\ (= 70)$ *A1*

$\mathrm{LB} = 70 + 18 + 22$ *(M1)*

$= 110$ *A1*

[4 marks]

(f) Explain, with a reason, why the answer to part (e) might not be the best lower bound. [2]

Markscheme

EITHER

deleting a different vertex ***A1***

might give a higher value (and hence a better lower bound). ***R1***

OR

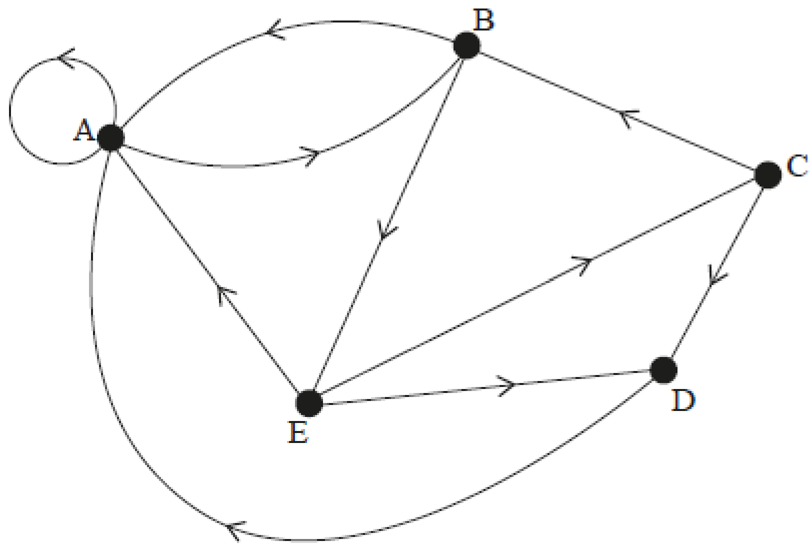
the edges selected in part (e) do not form a cycle. ***A1***

so a higher value is possible ***R1***

[2 marks]

4. [Maximum mark: 5] 22M.1.AHL.TZ2.6

Consider the following directed network.



(a) Write down the adjacency matrix for this network. [2]

Markscheme

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$ ***A2***

Note: Award ***A2*** for the transposed matrix. Presentation in markscheme assumes columns/rows ordered A-E; accept a matrix with rows and/or columns in a different order only if appropriately communicated. Do not ***FT*** from part (a) into part (b).

[2 marks]

(b) Determine the number of different walks of length 5 that start and end at the same vertex. [3]

Markscheme

raising their matrix to a power of 5 ***(M1)***

$$\boldsymbol{M}^5 = \begin{pmatrix} 17 & 9 & 2 & 3 & 5 \\ 17 & 10 & 3 & 4 & 4 \\ 13 & 6 & 2 & 2 & 4 \\ 8 & 5 & 1 & 2 & 2 \\ 18 & 11 & 2 & 4 & 5 \end{pmatrix}$$ ***(A1)***

Note: The numbers along the diagonal are sufficient to award ***M1A1***.

(the required number is $17 + 10 + 2 + 2 + 5 =$) 36 ***A1***

[3 marks]

5. [Maximum mark: 7] 21M.1.AHL.TZ1.10

An engineer plans to visit six oil rigs $(\mathrm{A–F})$ in the Gulf of Mexico, starting and finishing at A. The travelling time, in minutes, between each of the rigs is shown in the table.

	A	B	C	D	E	F
A		55	63	79	87	93
B	55		46	58	88	92
C	63	46		87	77	66
D	79	58	87		23	70
E	87	88	77	23		47
F	93	92	66	70	47	

The data above can be represented by a graph G.

(a.i) Use Prim's algorithm to find the weight of the minimum spanning tree of the subgraph of G obtained by deleting A and starting at B. List the order in which the edges are selected. [4]

Markscheme

use of Prim's algorithm ***M1***

$\mathrm{BC} \quad 46$ ***A1***

$\mathrm{BD} \quad 58$ ***A1***

$\mathrm{DE} \quad 23$

$\mathrm{EF} \quad 47$

Total 174 ***A1***

Note: Award ***M0A0A0A1*** for 174 without correct working e.g. use of Kruskal's, or with no working.
Award ***M1A0A0A1*** for 174 by using Prim's from an incorrect starting point.

[4 marks]

(a.ii) Hence find a lower bound for the travelling time needed to visit all the oil rigs. [2]

Markscheme

$\mathrm{AB} + \mathrm{AC} = 55 + 63 = 118$ ***(M1)***

$174 + 118 = 292$ minutes ***A1***

[2 marks]

(b) Describe how an improved lower bound might be found. [1]

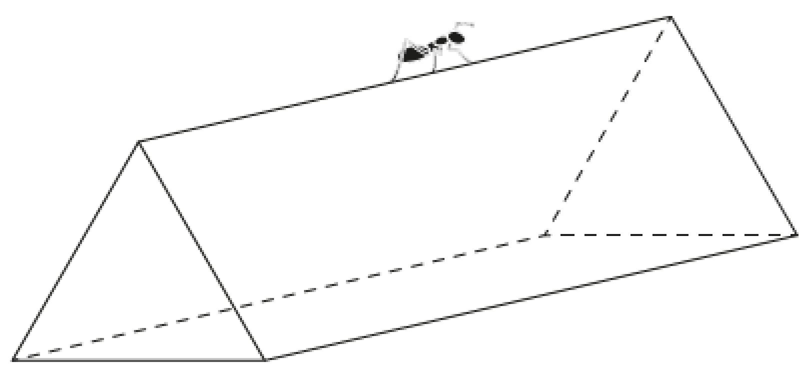
Markscheme

delete a different vertex ***A1***

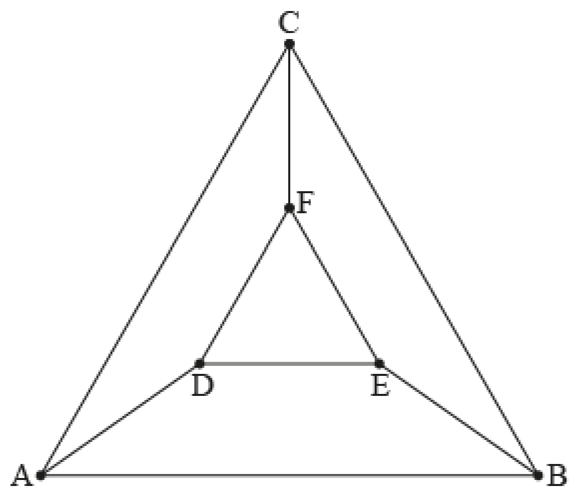
[1 mark]

6. [Maximum mark: 5] 21M.1.AHL.TZ1.16

An ant is walking along the edges of a wire frame in the shape of a triangular prism.



The vertices and edges of this frame can be represented by the graph below.



(a) Write down the adjacency matrix, $\boldsymbol{M}$, for this graph. [3]

Markscheme

$$\boldsymbol{M} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$ ***A1A1A1***

Note: Award ***A1*** for each two correct rows.

[3 marks]

(b) Find the number of ways that the ant can start at the vertex A, and walk along exactly 6 edges to return to A. [2]

Markscheme

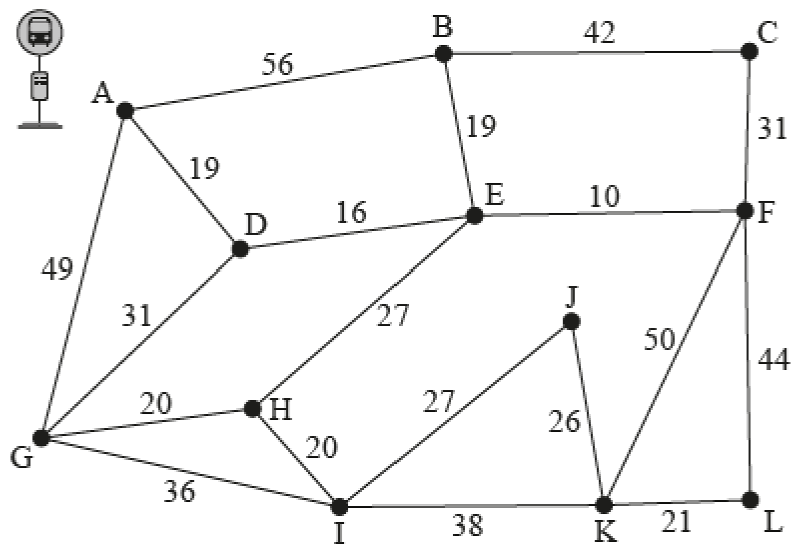
calculating $\boldsymbol{M}^6$ ***(M1)***

143 ***A1***

[2 marks]

7. [Maximum mark: 7] 21M.1.AHL.TZ2.11

The diagram below shows a network of roads in a small village with the weights indicating the distance of each road, in metres, and junctions indicated with letters.



Musab is required to deliver leaflets to every house on each road. He wishes to minimize his total distance.

(a) Musab starts and finishes from the village bus-stop at A.
Determine the total distance Musab will need to walk. [5]

Markscheme

Odd vertices are $\mathrm{A}, \mathrm{B}, \mathrm{D}, \mathrm{H}$ ***A1***

Consider pairings: ***M1***

Note: Award ***(M1)*** if there are four vertices not necessarily all correct.

$\mathrm{AB\ DH}$ has shortest route $\mathrm{AD, DE, EB}$ and $\mathrm{DE, EH,}$
so repeated edges $(19 + 16 + 19) + (16 + 27) = 97$

Note: Condone AB in place of $\mathrm{AD, DE, EB}$ giving
$56 + (16 + 27) = 99$.

$\mathrm{AD\ BH}$ has shortest route AD and $\mathrm{BE, EH,}$
so repeated edges $19 + (19 + 27) = 65$

$\mathrm{AH\ BD}$ has shortest route $\mathrm{AD, DE, EH}$ and $\mathrm{BE, ED}$,
so repeated edges $(19 + 16 + 27) + (19 + 16) = 97$ ***A2***

Note: Award ***A1*** if only one or two pairings are correctly considered.

so best pairing is $\mathrm{AD, BH}$
weight of route is therefore $582 + 65 = 647$ ***A1***

[5 marks]

(b) Instead of having to catch the bus to the village, Musab's sister offers to drop him off at any junction and pick him up at any other junction of his choice.

Explain which junctions Musab should choose as his starting and finishing points.

[2]

Markscheme

least value of the pairings is 19 therefore repeat AD ***R1***

B and H ***A1***

Note: Do not award ***R0A1***.

[2 marks]

8. [Maximum mark: 10] 19N.3.AHL.TZ0.Hdm_1

A driver needs to make deliveries to five shops A, B, C, D and E. The driver starts and finishes his journey at the warehouse W. The driver wants to find the shortest route to visit all the shops and return to the warehouse. The distances, in kilometres, between the locations are given in the following table.

	A	*B*	*C*	*D*	*E*	*W*
A	-	11	28	15	20	40
B	11	-	25	20	32	36
C	28	25	-	16	22	39
D	15	20	16	-	12	42
E	20	32	22	12	-	41
W	40	36	39	42	41	-

(a) By deleting W, use the deleted vertex algorithm to find a lower bound for the length of a route that visits every shop, starting and finishing at W. [6]

Markscheme

deleting W and its adjacent edges, the minimal spanning tree is

Edge	Weight
AB	11
DE	12
AD	15
CD	16

A1A1A1A1

Note: Award the ***A1's*** for either the edges or their weights.

the minimum spanning tree has weight = 54

Note: Accept a correct drawing of the minimal spanning tree.

adding in the weights of 2 deleted edges of least weight WB and WC
(M1)
lower bound = 54 + 36 + 39

= 129 ***A1***

[6 marks]

(b) Starting from W, use the nearest-neighbour algorithm to find a route which gives an upper bound for this problem and calculate its length. [4]

Markscheme

attempt at the nearest-neighbour algorithm ***M1***

WB
BA
AD
DE
EC
CW ***A1***

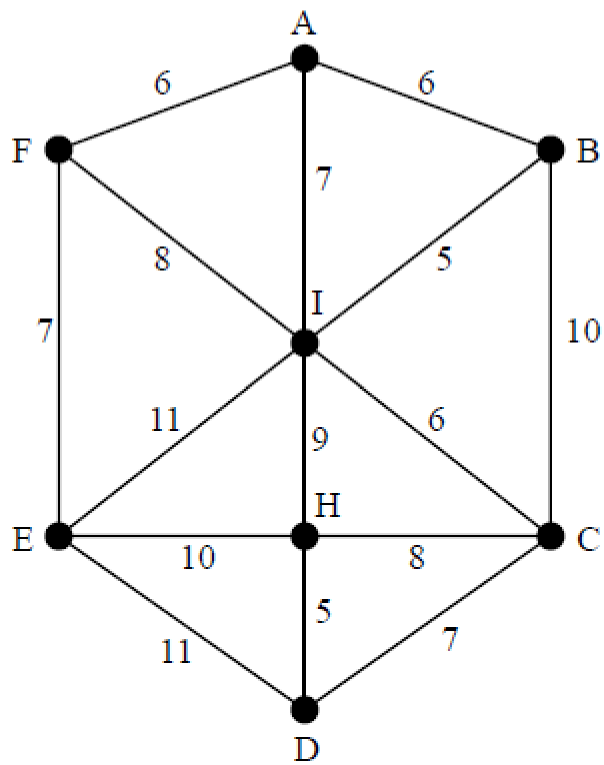
Note: Award ***M1*** for a route that begins with WB and then BA.

upper bound = 36 + 11 + 15 + 12 + 22 + 39 = 135 ***(M1)A1***

[4 marks]

9. [Maximum mark: 17] 18N.3.AHL.TZ0.Hdm_4

Consider the graph G represented in the following diagram.



(a) State, with a reason, whether or not G has an Eulerian circuit. [1]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

no because the graph has vertices (A, B, D, F) of odd degree ***R1***

[1 mark]

(b) Use Kruskal's algorithm to find a minimum spanning tree for G, stating its total weight. Indicate clearly the order in which the edges are added. [4]

Markscheme

the edges are added in the order

BI 5

DH 5 ***A1***

AB 6

AF 6

CI 6 ***A1***

CD 7

EF 7 ***A1***

total weight = 42 ***A1***

Note: The orders of the edges with the same weight are interchangeable. Accept indication of correct edge order on a diagram.

[4 marks]

The graph G is a plan of a holiday resort where each vertex represents a villa and the edges represent the roads between villas. The weights of the edges are the times, in minutes, Mr José, the security guard, takes to walk along each of the roads. Mr José is based at villa A.

(c) Use a suitable algorithm to show that the minimum time in which Mr José can get from A to E is 13 minutes. [5]

Markscheme

clear indication of using Dijkstra for example ***M1***

A	B	C	D	E	F	H	I	Vertex (time)	
	6(A)	-	-	-	6(A)	-	7(A)	A(0)	***A1***
		16(B)	-	-	6(A)	-	7(A)	B(6)	***A1***
		16(B)	-	13(F)		-	7(A)	F(6)	***A1***
		13(I)	-	13(F)		16(I)		I(7)	***A1***
								E(13)	***AG***

[5 marks]

(d) Find the minimum time it takes Mr José to patrol the resort if he has to walk along every road at least once, starting and ending at A. State clearly which roads need to be repeated. [7]

Markscheme

there are 4 vertices of odd degree (A, F, B and D) ***(A1)***

attempting to list at least 2 possible pairings of odd vertices ***M1***

A → F and B → D has minimum weight 6 + 17 = 23

A → B and F → D has minimum weight 6 + 18 = 24

A → D and F → B has minimum weight 20 + 12 = 32 ***A1A1***

Note: Award ***A1A0*** for 2 pairs.

minimum time is (116 + 23 =) 139 (mins) ***(M1)A1*** roads repeated are AF, BC and CD ***A1***

[7 marks]