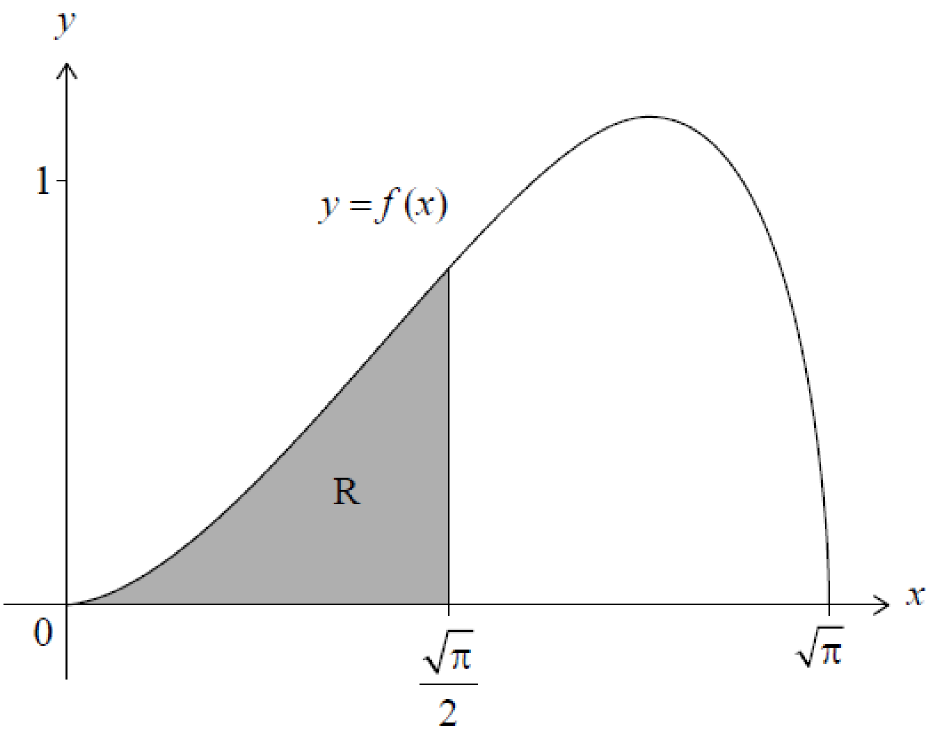
Definite integrals [47 marks]

1. [Maximum mark: 6] 24M.1.AHL.TZ1.6

The function f is defined as $f(x) = \sqrt{x \sin\left(x^2\right)}$, where $0 \leq x \leq \sqrt{\pi}$.

Consider the shaded region R enclosed by the graph of f, the x-axis and the line $x = \frac{\sqrt{\pi}}{2}$, as shown in the following diagram.



The shaded region R is rotated by 2π radians about the x-axis to form a solid.

Show that the volume of the solid is $\frac{\pi\left(2-\sqrt{2}\right)}{4}$. [6]

Markscheme

METHOD 1

attempt to find an integral involving π and the square of $f(x)$ ***M1***

Note: Condone incorrect or absent limits for this ***M1***.

$\pi \int_0^{\frac{\sqrt{\pi}}{2}} (f(x))^2 \text{ d } x$

$\pi \int_0^{\frac{\sqrt{x}}{2}} x \sin\left(x^2\right) \text{ d } x$ ***A1***

EITHER

attempt to use integration by substitution ***M1***

$\frac{\pi}{2} \int_0^{\frac{\pi}{4}} \sin(u) \text{ d } u$

Note: Award ***M1*** for $u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

$= \left[-\frac{\pi}{2} \cos(u)\right]_0^{\frac{\pi}{4}}$ ***A1***

OR

attempt to integrate by inspection ***(M1)***

$\frac{\pi}{2} \int_0^{\frac{\sqrt{\pi}}{2}} 2x \sin\left(x^2\right) \text{ d } x$ OR $\frac{\pi}{2} \int_0^{\frac{\sqrt{\pi}}{2}} \sin\left(x^2\right) \text{ d } \left(x^2\right)$

$= \left[-\frac{\pi}{2} \cos\left(x^2\right)\right]_0^{\frac{\sqrt{\pi}}{2}}$ ***A1***

Note: Condone incorrect or absent limits for ***M1***.

The correct limits may be seen or implied by later work for the ***A1***.

THEN

$= \left(-\frac{\pi}{2}\cos\left(\frac{\pi}{4}\right)\right) - \left(-\frac{\pi}{2}\cos(0)\right)$ (or equivalent) ***(A1)***

$= -\frac{\pi}{2\sqrt{2}} + \frac{\pi}{2}$ OR $-\frac{\pi\sqrt{2}}{4} + \frac{\pi}{2}$ OR $\frac{\pi}{2}\left(-\frac{1}{\sqrt{2}} + 1\right)$ OR $\frac{\pi}{2}\left(-\frac{\sqrt{2}}{2} + 1\right)$ ***A1***

$= \frac{\pi\left(2-\sqrt{2}\right)}{4}$ ***AG***

METHOD 2

attempt to find an integral involving π and the square of $f(x)$ ***M1***

Note: Condone incorrect or absent limits for this ***M1***.

$\pi\int_0^{\frac{\sqrt{\pi}}{2}} (f(x))^2 \, \mathrm{d}\,x$

$\pi\int_0^{\frac{\sqrt{\pi}}{2}} x\sin\left(x^2\right) \mathrm{d}\,x$ ***A1***

attempt to use integration by substitution ***M1***

$u = \cos\left(x^2\right) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -2x\sin\left(x^2\right)$

Note: Award ***M1*** for $u = \cos\left(x^2\right)$

$$= -\frac{\pi}{2}\int_{-\frac{1}{\sqrt{2}}}^{-1} \mathrm{d}\,u$$

$$= \left[-\frac{\pi}{2}u\right]_{-\frac{1}{\sqrt{2}}}^{-1}$$ (or equivalent) ***A1A1***

Note: Condone incorrect or absent limits for ***M1***.

A1 for $-\frac{\pi}{2}u$ and ***A1*** for both correct limits.

$$= \frac{\pi}{2} - \frac{\pi}{2\sqrt{2}} \text{ OR } \frac{\pi}{2} - \frac{\pi\sqrt{2}}{4} \text{ OR } \frac{\pi}{2}\left(1 - \frac{1}{\sqrt{2}}\right) \text{ OR } \frac{\pi}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$$

A1

$$= \frac{\pi\left(2-\sqrt{2}\right)}{4}$$ ***AG***

[6 marks]

2. [Maximum mark: 7] 23M.1.AHL.TZ1.9

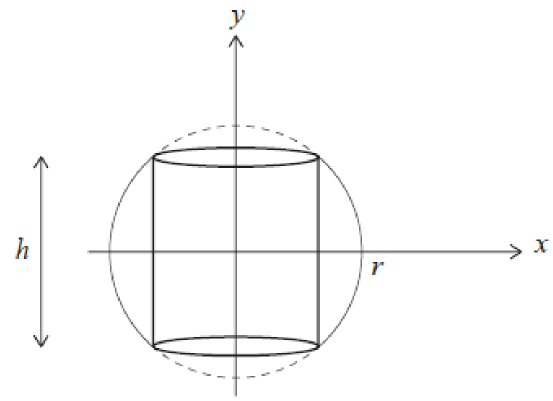
The function f is defined by $f\left(y\right) = \sqrt{r^2 - y^2}$ for $-r \le y \le r$

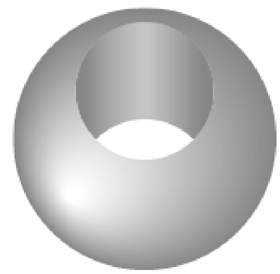
.

The region enclosed by the graph of $x = f(y)$ and the y-axis is rotated by $360\,^\circ$ about the y-axis to form a solid sphere. The sphere is drilled through along the y-axis, creating a cylindrical hole. The resulting spherical ring has height, h.

This information is shown in the following diagrams.

diagram not to scale





The spherical ring has a volume of π cubic units. Find the value of h. [7]

Markscheme

METHOD 1 (subtracting volumes)

radius of cylinder, R is $\sqrt{r^2 - \frac{h^2}{4}}$ OR $R^2 = r^2 - \frac{h^2}{4}$ (seen anywhere) ***(A1)***

correct limits 0 and $\frac{h}{2}$ OR $-\frac{h}{2}$ and $\frac{h}{2}$ (seen anywhere) ***(A1)***

EITHER

volume of part sphere $= \pi \int \left(r^2 - y^2\right) \mathrm{d}y$

correct integration ***A1***

$r^2 y - \frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{r^2 h}{2} - \frac{h^3}{24}$

recognition that the volume of the ring is $\pi\int\left(r^2-y^2\right)\mathrm{d}y-\pi R^2h$ where $R\neq r$ *(M1)*

$\pi\int\left(r^2-y^2\right)\mathrm{d}y-\pi\left(r^2-\frac{h^2}{4}\right)h$ (or equivalent)

correct equation *(A1)*

$2\pi\left(\frac{r^2h}{2}-\frac{h^3}{24}\right)-\pi r^2h+\frac{\pi h^3}{4}=\pi$ OR $\frac{h^3}{4}-\frac{h^3}{12}=1$ (or equivalent)

OR

recognition that the volume of the ring is
$\pi\int\left(\left(r^2-y^2\right)-\left(r^2-\frac{h^2}{4}\right)\right)\mathrm{d}y$ (or equivalent) *(M1)*

correct integration *A1*

$\frac{h^2}{4}y-\frac{y^3}{3}$

attempt to substitute their limits into their integrated expression *(M1)*

$\frac{h^3}{8}-\frac{h^3}{24}$

correct equation *(A1)*

$2\pi\left(\frac{h^3}{8}-\frac{h^3}{24}\right)=\pi$ OR $2\left(\frac{h^3}{8}-\frac{h^3}{24}\right)=1$ (or equivalent)

THEN

$h^3=\sqrt[3]{6}$ *A1*

METHOD 2 (volume of cylindrical hole)

radius of cylinder, R is $\sqrt{r^2-\frac{h^2}{4}}$ OR $R^2=r^2-\frac{h^2}{4}$ (seen anywhere) *(A1)* correct limits $\frac{h}{2}$ and r (seen anywhere) ***(A1)***

volume of part sphere $= \pi \int \left(r^2 - y^2\right) \mathrm{d}y$

correct integration ***A1***

$r^2 y - \frac{y^3}{3}$

attempt to substitute their limits into their integrated expression ***(M1)***

$\frac{2r^3}{3} - \frac{r^2 h}{2} + \frac{h^3}{24}$

recognition that the volume of the cylindrical hole is
$\pi \int \left(r^2 - y^2\right) \mathrm{d}y + \pi R^2 h$ where $R \neq r$ ***(M1)***

$\pi \int \left(r^2 - y^2\right) \mathrm{d}y + \pi \left(r^2 - \frac{h^2}{4}\right) h \left(= \frac{4}{3}\pi r^3 - \pi\right)$ (or equivalent)

correct equation ***(A1)***

$2\pi \left(\frac{2r^3}{3} - \frac{r^2 h}{2} + \frac{h^3}{24}\right) + \pi r^2 h - \frac{\pi h^3}{4} = \frac{4}{3}\pi r^3 - \pi$ OR
$\frac{h^3}{12} - \frac{h^3}{4} = -1$ (or equivalent)

$h = \sqrt[3]{6}$ ***A1***

METHOD 3 (shells)

radius of cylinder, R is $\sqrt{r^2 - \frac{h^2}{4}}$ OR $R^2 = r^2 - \frac{h^2}{4}$ (seen anywhere) ***(A1)***

attempt to use shells method ***(M1)***

$2\pi \int x\sqrt{r^2 - x^2} \mathrm{d}x$

correct limits r and $\sqrt{r^2-\frac{h^2}{4}}$ (seen anywhere) ***(A1)***

correct integration ***A1***

$-\frac{1}{3}\left(r^2-x^2\right)^{\frac{3}{2}}$

attempt to substitute their limits into their integrated expression ***(M1)***

$-\frac{1}{3}\left(0-\left(r^2-\left(r^2-\frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)$

correct equation ***(A1)***

$2\times\frac{-2\pi}{3}\left(0-\left(r^2-\left(r^2-\frac{h^2}{4}\right)\right)^{\frac{3}{2}}\right)=\pi$ OR

$2\left(\frac{2\pi}{3}\times\frac{h^3}{8}\right)=\pi$

$h=\sqrt[3]{6}$ ***A1***

[7 marks]

3. [Maximum mark: 5] 22M.1.AHL.TZ1.1

Find the value of $\int_1^9\left(\frac{3\sqrt{x}-5}{\sqrt{x}}\right)\mathrm{d}\,x$. [5]

Markscheme

$\int\frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}\,x=\int\left(3-5x^{-\frac{1}{2}}\right)\mathrm{d}\,x$ ***(A1)***

$\int\frac{3\sqrt{x}-5}{\sqrt{x}}\mathrm{d}\,x=3x-10x^{\frac{1}{2}}(+c)$ ***A1A1***

substituting limits into their integrated function and subtracting ***(M1)***

$3(9)-10(9)^{\frac{1}{2}}-\left(3(1)-10(1)^{\frac{1}{2}}\right)$ OR

$27-10\times 3-(3-10)$

$=4$ ***A1***

[5 marks]

4. [Maximum mark: 6] 22M.1.AHL.TZ2.7

By using the substitution $u=\sec x$ or otherwise, find an expression for $\int_0^{\frac{\pi}{3}} \sec^n x \tan x \, \mathrm{d}\, x$ in terms of n, where n is a non-zero real number. [6]

Markscheme

METHOD 1

$u=\sec x \Rightarrow \mathrm{d}\, u=\sec x \tan x \, \mathrm{d}\, x$ ***(A1)***

attempts to express the integral in terms of u ***M1***

$\int_1^2 u^{n-1} \, \mathrm{d}\, u$ ***A1***

$=\frac{1}{n}[u^n]_1^2 \left(=\frac{1}{n}[\sec^n x]_0^{\frac{\pi}{3}}\right)$ ***A1***

Note: Condone the absence of or incorrect limits up to this point.

$=\frac{2^n-1^n}{n}$ ***M1***

$= \frac{2^n - 1}{n}$ ***A1***

Note: Award ***M1*** for correct substitution of **<u>their</u>** limits for u into their antiderivative for u (or given limits for x into their antiderivative for x).

METHOD 2

$\int \sec^n x \tan x \, \mathrm{d}\, x = \int \sec^{n-1} x \sec x \tan x \, \mathrm{d}\, x$ ***(A1)***

applies integration by inspection ***(M1)***

$= \frac{1}{n} [\sec^n x]_0^{\frac{\pi}{3}}$ ***A2***

Note: Award ***A2*** if the limits are not stated.

$= \frac{1}{n} \left(\sec^n \frac{\pi}{3} - \sec^n 0 \right)$ ***M1***

Note: Award ***M1*** for correct substitution into their antiderivative.

$= \frac{2^n - 1}{n}$ ***A1***

[6 marks]

5. [Maximum mark: 6] 19N.1.AHL.TZ0.H_2

Given that $\int_0^{\ln k} \mathrm{e}^{2x} \mathrm{d}x = 12$, find the value of k. [6]

Markscheme

$\frac{1}{2}\mathrm{e}^{2x}$ seen ***(A1)***

attempt at using limits in an integrated expression $\left(\left[\frac{1}{2}\mathrm{e}^{2x}\right]_0^{\ln k} = \frac{1}{2}\mathrm{e}^{2\ln k} - \frac{1}{2}\mathrm{e}^{0}\right)$ ***(M1)***

$= \frac{1}{2}\mathrm{e}^{\ln k^2} - \frac{1}{2}\mathrm{e}^{0}$ ***(A1)***

Setting their equation $= 12$ ***M1***

Note: their equation must be an integrated expression with limits substituted.

$\frac{1}{2}k^2 - \frac{1}{2} = 12$ ***A1***

$\left(k^2 = 25 \Rightarrow\right) k = 5$ ***A1***

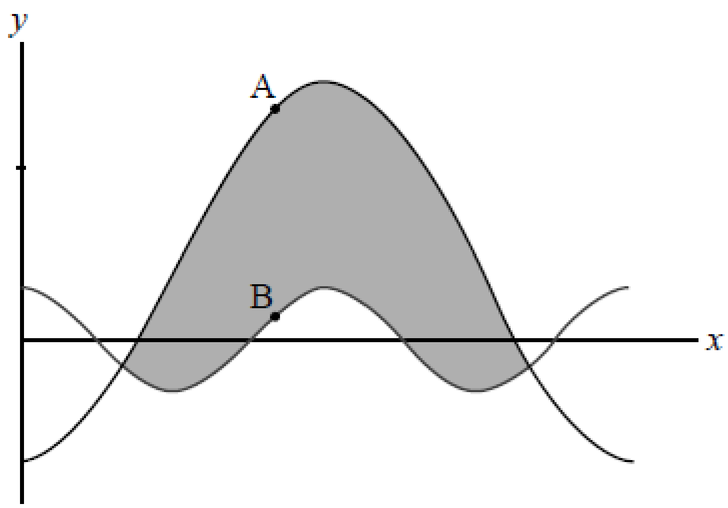
Note: Do not award final ***A1*** for $k = \pm 5$.

[6 marks]

6. [Maximum mark: 17] 19M.1.AHL.TZ2.H_9

Consider the functions f and g defined on the domain $0 < x < 2\pi$ by $f(x) = 3\cos 2x$ and $g(x) = 4 - 11\cos x$.

The following diagram shows the graphs of $y = f(x)$ and $y = g(x)$



(a) Find the x-coordinates of the points of intersection of the two graphs. [6]

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$$3\cos 2x = 4 - 11\cos x$$

attempt to form a quadratic in $\cos x$ ***M1***

$$3\left(2\cos^2 x - 1\right) = 4 - 11\cos x$$ ***A1***

$$\left(6\cos^2 x + 11\cos x - 7 = 0\right)$$

valid attempt to solve their quadratic ***M1***

$$(3\cos x + 7)(2\cos x - 1) = 0$$

$\cos x = \frac{1}{2}$ ***A1***

$x = \frac{\pi}{3}, \frac{5\pi}{3}$ ***A1A1***

Note: Ignore any "extra" solutions.

[6 marks]

(b) Find the exact area of the shaded region, giving your answer in the form $p\pi + q\sqrt{3}$, where $p, q \in \mathbb{Q}$. [5]

Markscheme

consider (±) $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (4 - 11\cos x - 3\cos 2x)\,\mathrm{d}x$ ***M1***

$= (\pm)\left[4x - 11\sin x - \frac{3}{2}\sin 2x\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$ ***A1***

Note: Ignore lack of or incorrect limits at this stage.

attempt to substitute their limits into their integral ***M1***

$= \frac{20\pi}{3} - 11\sin\frac{5\pi}{3} - \frac{3}{2}\sin\frac{10\pi}{3} - \left(\frac{4\pi}{3} - 11\sin\frac{\pi}{3} - \frac{3}{2}\sin\frac{2\pi}{3}\right)$

$= \frac{16\pi}{3} + \frac{11\sqrt{3}}{2} + \frac{3\sqrt{3}}{4} + \frac{11\sqrt{3}}{2} + \frac{3\sqrt{3}}{4}$

$= \frac{16\pi}{3} + \frac{25\sqrt{3}}{2}$ ***A1A1***

[5 marks]

(c) At the points A and B on the diagram, the gradients of the two graphs are equal.

Determine the y-coordinate of A on the graph of g. [6]

Markscheme

attempt to differentiate both functions and equate ***M1***

$-6\sin 2x = 11\sin x$ ***A1***

attempt to solve for x ***M1***

$11\sin x + 12\sin x\cos x = 0$

$\sin x\,(11 + 12\cos x) = 0$

$\cos x = -\frac{11}{12}$ (or $\sin x = 0$) ***A1***

$\Rightarrow y = 4 - 11\left(-\frac{11}{12}\right)$ ***M1***

$y = \frac{169}{12}\left(= 14\frac{1}{12}\right)$ ***A1***

[6 marks]