Functions, exponents and logarithms (GDC) [44 marks]

1. [Maximum mark: 17] EXN.2.SL.TZ0.9

The temperature $T\ ^\circ\mathrm{C}$ of water t minutes after being poured into a cup can be modelled by $T = T_0\mathrm{e}^{-kt}$ where $t \geq 0$ and $T_0,\ k$ are positive constants.

The water is initially boiling at $100\ ^\circ\mathrm{C}$. When $t = 10$, the temperature of the water is $70\ ^\circ\mathrm{C}$.

(a) Show that $T_0 = 100$. [1]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

when $t = 0,\ T = 100 \Rightarrow 100 = T_0\mathrm{e}^0$ **A1**

so $T_0 = 100$ **AG**

[1 mark]

(b) Show that $k = \frac{1}{10}\ln\frac{10}{7}$. [3]

Markscheme

correct substitution of $t = 10,\ T = 70$ **M1**

$70 = 100\mathrm{e}^{-10k}$ or $\mathrm{e}^{-10k} = \frac{7}{10}$

EITHER

$-10k = \ln\frac{7}{10}$ **A1**

$\ln\frac{7}{10} = -\ln\frac{10}{7}$ or $-\ln\frac{7}{10} = \ln\frac{10}{7}$ **A1**

OR

$\mathrm{e}^{10k} = \frac{10}{7}$ **A1**

$10k = \ln\frac{10}{7}$ **A1**

THEN

$k = \frac{1}{10}\ln\frac{10}{7}$ **AG**

[3 marks]

(c) Find the temperature of the water when $t = 15$. [2]

Markscheme

substitutes $t = 15$ into T **(M1)**

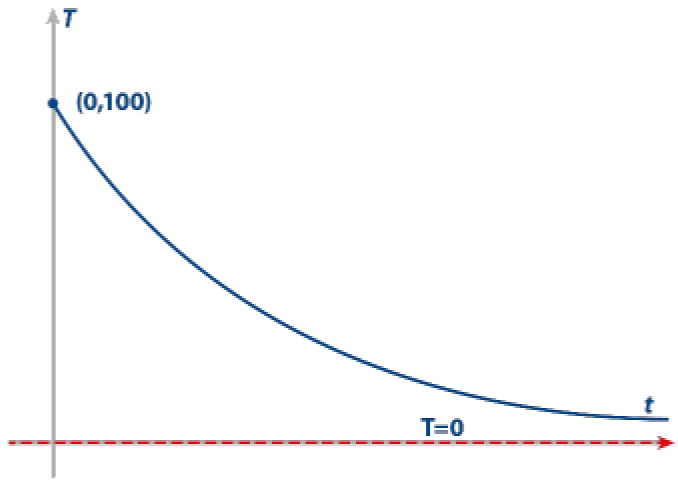
$T = 58.6\,(^\circ\mathrm{C})$ **A1**

[2 marks]

(d) Sketch the graph of T versus t, clearly indicating any asymptotes with their equations and stating the coordinates of any points of intersection with the axes.

[4]

Markscheme



a decreasing exponential **A1**

starting at $(0,\ 100)$ labelled on the graph or stated **A1**

$T \to 0$ as $t \to \infty$ **A1**

horizontal asymptote $T = 0$ labelled on the graph or stated **A1**

Note: Award **A0** for stating $y = 0$ as the horizontal asymptote.

[4 marks]

(e) Find the time taken for the water to have a temperature of $50\ ^\circ\mathrm{C}$. Give your answer correct to the nearest second. [4]

Markscheme

$100\mathrm{e}^{-kt} = 50$ where $k = \frac{1}{10}\ln\frac{10}{7}$ **A1**

EITHER

uses an appropriate graph to attempt to solve for t **(M1)**

OR

manipulates logs to attempt to solve for t e.g. $\ln\frac{1}{2}=\left(-\frac{1}{10}\ln\frac{10}{7}\right)t$ **(M1)**

$t=\frac{\ln 2}{\frac{1}{10}\ln\frac{10}{7}}=19.433\ldots$ **A1**

THEN

temperature will be $50\ ^\circ\mathrm{C}$ after 19 minutes and 26 seconds **A1**

[4 marks]

(f) The model for the temperature of the water can also be expressed in the form $T=T_0a^{\frac{t}{10}}$ for $t\geq 0$ and a is a positive constant.

Find the exact value of a. [3]

Markscheme

METHOD 1

substitutes $T_0=100, t=10$ and $T=70$ into $T=T_0a^{\frac{t}{10}}$ **(M1)**

$70=100a^{\frac{10}{10}}$ **A1**

$a = \frac{7}{10}$ **A1**

METHOD 2

$100a^{\frac{t}{10}} = 100\mathrm{e}^{-kt}$ where $k = \frac{1}{10}\ln\frac{10}{7}$

EITHER

$\mathrm{e}^{-k} = a^{\frac{1}{10}} \Rightarrow a = \mathrm{e}^{-10k}$ **(M1)**

OR

$a = \left(\mathrm{e}^{\left(-\frac{1}{10}\ln\frac{10}{7}\right)t}\right)^{\frac{10}{t}}$ **(M1)**

THEN

$a = \mathrm{e}^{-\ln\frac{10}{7}} \left(= \mathrm{e}^{\ln\frac{7}{10}}\right)$ **A1**

$a = \frac{7}{10}$ **A1**

[3 marks]

2. [Maximum mark: 6] 24M.2.SL.TZ2.4

The loudness of a sound, L, measured in decibels, is related to its intensity, I units, by $L = 10\log_{10}\left(I \times 10^{12}\right)$.

Consider two sounds, S_1 and S_2.

S_1 has an intensity of 10^{-6} units and a loudness of 60 decibels.

S_2 has an intensity that is twice that of S_1.

(a) State the intensity of S_2. [1]

Markscheme

$I = 2 \times 10^{-6} \left(= \frac{1}{500\,000}\right)$ (units) ***A1***

[1 mark]

(b) Determine the loudness of S_2. [2]

Markscheme

substitutes their doubled I-value from part (a) into L ***(M1)***

$L = 10 \log_{10} \left(2 \times 10^{-6} \times 10^{12}\right) \ (= 63.0102\ldots)$

$= 63.0$ (decibels) ***A1***

Note: Accept $60 + 10 \log_{10} 2$ (decibels) as a final answer.

Do not award the final ***A1*** for $L = 0$ (from $I = 10^{-12}$).

[2 marks]

The maximum loudness of thunder in a thunderstorm was measured to be 115 decibels.

(c) Find the corresponding intensity, I, of the thunder. [3]

Markscheme

$115 = 10 \log_{10} \left(I \times 10^{12}\right)$ ***(A1)***

attempts to solve for I ***(M1)***

$I = \frac{10^{11.5}}{10^{12}}$ (or equivalent) $(= 0.316227\ldots)$

$I = 0.316$ (units) ***A1***

Note: Accept exact final answers such as $10^{-0.5}$ and $\frac{1}{\sqrt{10}}$.

[3 marks]

3. [Maximum mark: 7] 23M.2.SL.TZ1.7

The temperature of a cup of tea, t minutes after it is poured, can be modelled by $H\left(t\right) = 21 + 75\mathrm{e}^{-0.08t},\ t \geq 0$. The temperature is measured in degrees Celsius ($^{\circ}\mathrm{C}$).

(a.i) Find the initial temperature of the tea. [1]

Markscheme

$96\left(^{\circ}\right)$ (exact) ***A1***

[1 mark]

(a.ii) Find the temperature of the tea three minutes after it is poured. [1]

Markscheme

$79.9970\ldots$

$80.0(^\circ)$ (accept 80) ***A1***

[1 mark]

(b) After k minutes, the tea will be below $67\ ^\circ\mathrm{C}$ and cool enough to drink.

Find the least possible value of k, where $k \in \mathbb{Z}^+$. [3]

Markscheme

METHOD 1

valid attempt to solve $H(t) = 67$ (accept an inequality) ***(M1)***

eg intersection of graphs, use of logarithms.

6.11058… ***(A1)***

7 (min) ***A1***

METHOD 2

valid attempt to find crossover values ***(M1)***

(6, 67.4087…) and (7, 63.8406…) ***(A1)***

7 (min) ***A1***

[3 marks]

As the tea cools, $H(t)$ approaches the temperature of the room, which is constant.

(c) Find the temperature of the room. [2]

Markscheme

recognition that $t \to \infty$ ***(M1)***

$21\ (^\circ\mathrm{C})$ ***A1***

[2 marks]

4. [Maximum mark: 7] 22N.2.SL.TZ0.5

The population of a town t years after 1 January 2014 can be modelled by the function

$P(t) = 15\,000\mathrm{e}^{kt}$, where $k < 0$ and $t \geq 0$.

It is known that between 1 January 2014 and 1 January 2022 the population decreased by 11%.

Use this model to estimate the population of this town on 1 January 2041. [7]

Markscheme

recognition that initial population is 15000 (seen anywhere) ***(A1)***

$P(0) = 15000$ OR 0.11×15000 OR 0.89×15000

population after 11% decrease is $15000 \times 0.89 (= 13350)$ ***(A1)***

recognizing that $t = 8$ on 1 January 2022 (seen anywhere) ***(A1)***

substitution of their value of t for 1 January 2022 and their value of $P(8)$ into the model ***(M1)***

$15000 \times 0.89 = 15000e^{8k}$ OR $13350 = 15000e^{8k}$

$k = \frac{\ln 0.89}{8}(-0.014566)$ ***(A1)***

substitution of $t = 2041 - 2014(= 27)$ and their value for k into the model ***(M1)***

$P(27) = 15000e^{-0.0145\ldots \times 27}$

$10122.3\ldots$

$P(27) = 10100\ (10122)$ ***A1***

[7 marks]

5. [Maximum mark: 7] 21M.2.SL.TZ2.6

All living plants contain an isotope of carbon called carbon-14. When a plant dies, the isotope decays so that the amount of carbon-14 present in the remains of the plant decreases. The time since the death of a plant can be determined by measuring the amount of carbon-14 still present in the remains.

The amount, A, of carbon-14 present in a plant t years after its death can be modelled by $A = A_0 e^{-kt}$ where $t \geq 0$ and $A_0,\ k$ are positive constants.

At the time of death, a plant is defined to have 100 units of carbon-14.

(a) Show that $A_0 = 100$. [1]

Markscheme

$100 = A_0 e^0$ ***A1***

$A_0 = 100$ ***AG***

[1 mark]

The time taken for half the original amount of carbon-14 to decay is known to be 5730 years.

(b) Show that $k = \frac{\ln 2}{5730}$. [3]

Markscheme

correct substitution of values into exponential equation ***(M1)***

$50 = 100e^{-5730k}$ OR $e^{-5730k} = \frac{1}{2}$

EITHER

$-5730k = \ln \frac{1}{2}$ ***A1***

$\ln \frac{1}{2} = -\ln 2$ OR $-\ln \frac{1}{2} = \ln 2$ ***A1***

OR

$e^{5730k} = 2$ ***A1***

$5730k = \ln 2$ ***A1***

THEN

$k = \frac{\ln 2}{5730}$ ***AG***

Note: There are many different ways of showing that $k = \frac{\ln 2}{5730}$ which involve showing different steps. Award full marks for at least two correct algebraic steps seen.

[3 marks]

(c) Find, correct to the nearest 10 years, the time taken after the plant's death for 25% of the carbon-14 to decay. [3]

Markscheme

if 25% of the carbon-14 has decayed, 75% remains ie, 75 units remain *(A1)*

$$75 = 100e^{-\frac{\ln 2}{5730}t}$$

EITHER

using an appropriate graph to attempt to solve for t *(M1)*

OR

manipulating logs to attempt to solve for t *(M1)*

$$\ln 0.75 = -\frac{\ln 2}{5730}t$$

$$t = 2378.164\ldots$$

THEN

$t = 2380$ (years) (correct to the nearest 10 years) *A1*

[3 marks]