Functions, exponents and logarithms (no GDC) [69 marks]

1. [Maximum mark: 5] 24M.1.SL.TZ1.3

It is given that $\log_{10} a = \frac{1}{3}$, where $a > 0$.

Find the value of

(a) $\log_{10}\left(\frac{1}{a}\right)$; [2]

Markscheme

$\log_{10} 1 - \log_{10} a$ OR $\log_{10} a^{-1} = -\log_{10} a$ OR $\log_{10} 10^{-\frac{1}{3}}$ OR $10^x = \frac{1}{10^{\frac{1}{3}}}$ ***(A1)***

$= -\frac{1}{3}$ ***A1***

[2 marks]

(b) $\log_{1000} a$. [3]

Markscheme

$\frac{\log_{10} a}{\log_{10} 1000}$ OR $\frac{1}{3}\log_{1000} 10$ OR $\log_{1000} \sqrt[3]{1000^{\frac{1}{3}}}$ OR $10^{\frac{1}{3}} = 1000^x \left(= \left(10^3\right)^x\right)$ ***(A1)***

$\frac{\log_{10} a}{3}$ OR $\frac{1}{3}\log_{1000} 1000^{\frac{1}{3}}$ OR $\log_{1000} 1000^{\frac{1}{9}}$ OR $3x = \frac{1}{3}$ ***(A1)***

$= \frac{1}{9}$ ***A1***

[3 marks]

2. [Maximum mark: 5] EXN.1.SL.TZ0.2

Solve the equation $2\ \ln\ x = \ \ln\ 9 + 4$. Give your answer in the form $x = p\text{e}^q$ where $p,\ q \in \mathbb{Z}^+$. [5]

Markscheme

* This sample question was produced by experienced DP mathematics senior examiners to aid teachers in preparing for external assessment in the new MAA course. There may be minor differences in formatting compared to formal exam papers.

METHOD 1

$2\ \ln\ x - \ \ln\ 9 = 4$

uses $m\ \ln\ x = \ \ln\ x^m$ **(M1)**

$\ln\ x^2 - \ \ln\ 9 = 4$

uses $\ln\ a - \ \ln\ b = \ln\ \frac{a}{b}$ **(M1)**

$\ln\ \frac{x^2}{9} = 4$

$\frac{x^2}{9} = \text{e}^4$ **A1**

$x^2 = 9\text{e}^4 \Rightarrow x = \sqrt{9\text{e}^4}\ \ (x > 0)$ **A1**

$x = 3\text{e}^2\ \ (p = 3,\ q = 2)$ **A1**

METHOD 2

expresses 4 as $4\ \ln\ \text{e}$ and uses $\ln\ x^m = m\ \ln\ x$ **(M1)**

$2\ \ln\ x = 2\ \ln\ 3 + 4\ \ln\ \text{e}\ \ (\ln\ x = \ln\ 3 + 2\ \ln\ \text{e})$ **A1**

uses $2\ \ln\ \text{e} = \ln\ \text{e}^2$ and $\ln\ a + \ln\ b = \ln\ ab$ **(M1)**

$\ln x = \ln \left(3e^2\right)$ **A1**

$x = 3e^2 \ (p = 3,\ q = 2)$ **A1**

METHOD 3

expresses 4 as $4 \ln e$ and uses $m \ln x = \ln x^m$ **(M1)**

$\ln x^2 = \ln 3^2 + \ln e^4$ **A1**

uses $\ln a + \ln b = \ln ab$ **(M1)**

$\ln x^2 = \ln \left(3^2 e^4\right)$

$x^2 = 3^2 e^4 \Rightarrow x = \sqrt{3^2 e^4} \ (x > 0)$ **A1**

so $x = 3e^2 \ (x > 0) \ (p = 3,\ q = 2)$ **A1**

[5 marks]

3. [Maximum mark: 6] EXN.1.SL.TZ0.5

The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = x - 2$ and $g(x) = ax + b$, where $a,\ b \in \mathbb{R}$.

Given that $(f \circ g)(2) = -3$ and $(g \circ f)(1) = 5$, find the value of a and the value of b. [6]

Markscheme

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new MAA course. There may be minor differences in formatting compared to formal exam papers.

$(f \circ g)(x) = ax + b - 2$ **(M1)**

$(f \circ g)(2) = -3 \Rightarrow 2a + b - 2 = -3 \ (2a + b = -1)$ **A1**

$(g \circ f)(x) = a(x - 2) + b$ **(M1)**

$(g \circ f)(1) = 5 \Rightarrow -a + b = 5$ **A1**

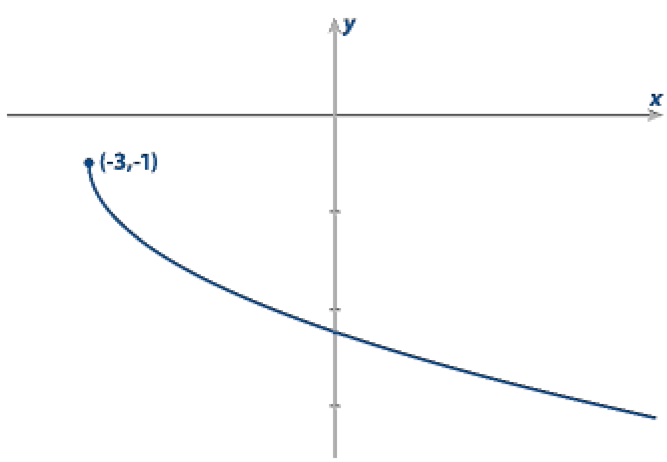
a valid attempt to solve their two linear equations for a and b **M1**

so $a = -2$ and $b = 3$ **A1**

[6 marks]

4. [Maximum mark: 14] EXN.1.SL.TZ0.8

The following diagram shows the graph of $y = -1 - \sqrt{x+3}$ for $x \geq -3$.



(a) Describe a sequence of transformations that transforms the graph of $y = \sqrt{x}$ for $x \geq 0$ to the graph of $y = -1 - \sqrt{x+3}$ for $x \geq -3$. [3]

Markscheme

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for example,

a reflection in the x-axis (in the line $y = 0$) **A1**

a horizontal translation (shift) 3 units to the left **A1**

a vertical translation (shift) down by 1 unit **A1**

Note: Award **A1** for each correct transformation applied in a correct position in the sequence. Do not accept use of the "move" for a translation.

Note: Award **A1A1A1** for a correct alternative sequence of transformations. For example,

a vertical translation (shift) down by 1 unit, followed by a horizontal translation (shift) 3 units to the left and then a reflection in the line $y = -1$.

[3 marks]

A function f is defined by $f(x) = -1 - \sqrt{x + 3}$ for $x \geq -3$.

(b) State the range of f. [1]

Markscheme

range is $f(x) \leq -1$ **A1**

Note: Correct alternative notations include $]-\infty, -1]$, $(-\infty, -1]$ or $y \leq -1$.

[1 mark]

(c) Find an expression for $f^{-1}(x)$, stating its domain. [5]

Markscheme

$-1-\sqrt{y+3}=x$ **M1**

Note: Award **M1** for interchanging x and y (can be done at a later stage).

$\sqrt{y+3}=-x-1(=-(x+1))$ **A1**

$y+3=(x+1)^2$ **A1**

so $f^{-1}(x)=(x+1)^2-3$ $\left(f^{-1}(x)=x^2+2x-2\right)$ **A1**

domain is $x \leq -1$ **A1**

Note: Correct alternative notations include $]-\infty, -1]$ or $(-\infty, -1]$.

[5 marks]

(d) Find the coordinates of the point(s) where the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect. [5]

Markscheme

the point of intersection lies on the line $y = x$

EITHER

$(x+1)^2 - 3 = x$ **M1**

attempts to solve their quadratic equation **M1**

for example, $(x+2)(x-1) = 0$ or $x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2}$ $\left(x = \frac{-1 \pm 3}{2}\right)$

OR

$-1 - \sqrt{x+3} = x$ **M1**

$\left(-1 - \sqrt{x+3}\right)^2 = x^2 \Rightarrow 2\sqrt{x+3} + x + 4 = x^2$

substitutes $2\sqrt{x+3} = -2(x+1)$ to obtain $-2(x+1) + x + 4 = x^2$

attempts to solve their quadratic equation **M1**

for example, $(x+2)(x-1) = 0$ or $x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2}$ $\left(x = \frac{-1 \pm 3}{2}\right)$

THEN

$x = -2, 1$ **A1**

as $x \le -1$, the only solution is $x = -2$ **R1**

so the coordinates of the point of intersection are $(-2,\ -2)$ **A1**

Note: Award **R0A1** if $(-2,\ -2)$ is stated without a valid reason given for rejecting $(1,\ 1)$.

[5 marks]

5. [Maximum mark: 5] 24M.1.AHL.TZ2.2

Solve $3 \times 9^x + 5 \times 3^x - 2 = 0$. [5]

Markscheme

recognising a quadratic in 3^x ***(M1)***

$3 \times (3^x)^2 + 5 \times 3^x - 2 = 0$

valid attempt to solve a quadratic equation (factorising, use of formula, completing square, or otherwise) ***(M1)***

$(3 \times 3^x - 1)(3^x + 2) = 0$ OR $3^x = \frac{-5 \pm \sqrt{25+24}}{6}$ (or equivalent) ***(A1)***

$3^x = \frac{1}{3}$ (or $3^x = -2$) ***(A1)***

$x = -1$ ***A1***

Note: Award the final ***A1*** if candidate's answer includes $x = -1$ and $x = \log_3 (-2)$. Award ***A0*** if other incorrect answers are given.

[5 marks]

6. [Maximum mark: 7] 23M.1.SL.TZ1.2

The function f is defined by $f(x) = \frac{7x+7}{2x-4}$ for $x \in \mathbb{R}, x \neq 2$.

(a) Find the zero of $f(x)$. [2]

Markscheme

recognizing $f(x) = 0$ ***(M1)***

$x = -1$ ***A1***

[2 marks]

(b) For the graph of $y = f(x)$, write down the equation of

(b.i) the vertical asymptote; [1]

Markscheme

$x = 2$ (must be an equation with x) ***A1***

[1 mark]

(b.ii) the horizontal asymptote. [1]

Markscheme

$y = \frac{7}{2}$ (must be an equation with y) ***A1***

[1 mark]

(c) Find $f^{-1}(x)$, the inverse function of $f(x)$. [3]

Markscheme

EITHER

interchanging x and y ***(M1)***

$2xy - 4x = 7y + 7$

correct working with y terms on the same side: $2xy - 7y = 4x + 7$
(A1)

OR

$2yx - 4y = 7x + 7$

correct working with x terms on the same side: $2yx - 7x = 4y + 7$
(A1)

interchanging x and y OR making x the subject $x = \frac{4y+7}{2y-7}$ ***(M1)***

THEN

$f^{-1}(x) = \frac{4x+7}{2x-7}$ (or equivalent) $\left(x \neq \frac{7}{2}\right)$ ***A1***

[3 marks]

7. [Maximum mark: 7] 23M.1.SL.TZ2.6

The functions f and g are defined for $x \in \mathbb{R}$ by

$f(x) = ax + b$, where $a, b \in \mathbb{Z}$

$g(x) = x^2 + x + 3$.

Find the two possible functions f such that $(g \circ f)(x) = 4x^2 - 14x + 15$.

[7]

Markscheme

attempts to form $(g \circ f)(x)$ ***(M1)***

$[f(x)]^2 + f(x) + 3$ OR $(ax + b)^2 + ax + b + 3$

$a^2x^2 + 2abx + b^2 + ax + b + 3 (= 4x^2 - 14x + 15)$ ***(A1)***

equates their corresponding terms to form at least one equation ***(M1)***

$a^2x^2 = 4x^2$ OR $a^2 = 4$ OR $2abx + ax = -14x$ OR
$2ab + a = -14$ OR $b^2 + b + 3 = 15$

$a = \pm 2$ (seen anywhere) ***A1***

attempt to use $2ab + a = -14$ to pair the correct values (seen anywhere) ***(M1)***

$f(x) = 2x - 4$ (accept $a = 2$ with $b = -4$), $f(x) = -2x + 3$ (accept $a = -2$ with $b = 3$) ***A1A1***

[7 marks]

8. [Maximum mark: 5] 23M.1.SL.TZ2.3

A function f is defined by $f(x) = 1 - \frac{1}{x-2}$, where $x \in \mathbb{R}, x \neq 2$.

(a) The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

Write down the equation of

(a.i) the vertical asymptote; [1]

Markscheme

$x = 2$ ***A1***

[1 mark]

(a.ii) the horizontal asymptote. [1]

Markscheme

$y = 1$ ***A1***

[1 mark]

(b) Find the coordinates of the point where the graph of $y = f(x)$ intersects

(b.i) the y-axis; [1]

Markscheme

$\left(0, \frac{3}{2}\right)$ ***A1***

[1 mark]

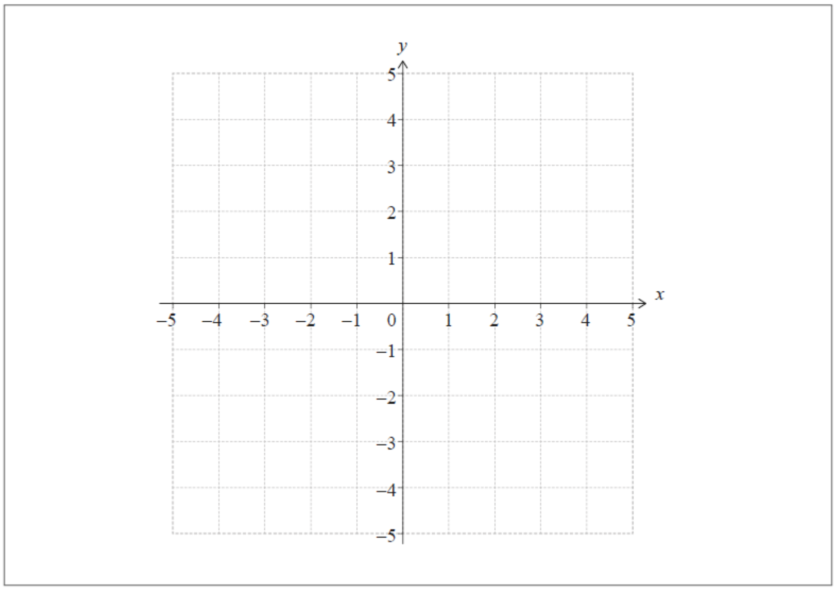
(b.ii) the x-axis. [1]

Markscheme

$(3, 0)$ ***A1***

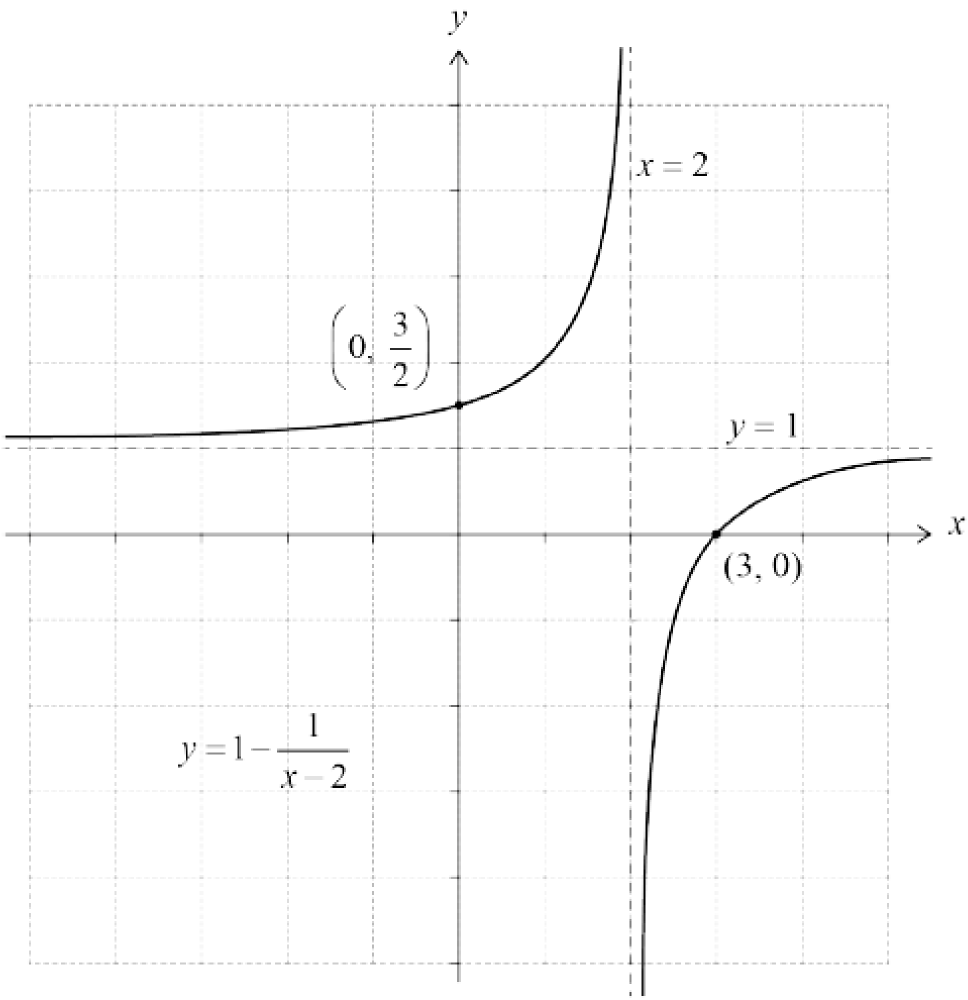
[1 mark]

(c) On the following set of axes, sketch the graph of $y = f(x)$, showing all the features found in parts (a) and (b).



[1]

Markscheme



two correct branches with correct asymptotic behaviour and intercepts clearly shown ***A1***

[1 mark]

9. [Maximum mark: 15] 22N.1.SL.TZ0.8

Calculate the value of each of the following logarithms:

(a.i) $\log_2 \frac{1}{16}$. [2]

Markscheme

valid approach to find the required logarithm ***(M1)***

$2^x = \frac{1}{16}$ OR $2^x = 2^{-4}$ OR $\frac{1}{16} = 2^{-4}$ OR $\log_2 1 - \log_2 16$

$\log_2 \frac{1}{16} = -4$ ***A1***

[2 marks]

(a.ii) $\log_9 3$. [2]

Markscheme

valid approach to find the required logarithm ***(M1)***

$9^x = 3$ OR $3^{2x} = 3$ OR $3 = 9^{\frac{1}{2}}$ OR $\frac{\log_3 3}{\log_3 9}$

$\log_9 3 = \frac{1}{2}$ ***A1***

[2 marks]

(a.iii) $\log_{\sqrt{3}} 81$. [3]

Markscheme

$\left(\sqrt{3}\right)^x = 81$ OR $\frac{\log_3 81}{\log_3 \sqrt{3}}$ ***(A1)***

$(3)^{\frac{x}{2}} = 3^4$ OR $\frac{x}{2} = 4$ OR $\frac{4}{\frac{1}{2}}$ ***(A1)***

$x = 8$ ***A1***

[3 marks]

It is given that $\log_{ab} a = 3$, where $a,\ b \in \mathbb{R}^+,\ ab \neq 1$.

(b.i) Show that $\log_{ab} b = -2$. [4]

Markscheme

Note: There are many valid approaches to the question, and the steps may be seen in different ways. Some possible methods are given here, but candidates may use a combination of one or more of these methods.

In all methods, the final ***A*** mark is awarded for working which leads directly to the ***AG***.

METHOD 1

$(ab)^3 = a$ ***(A1)***

attempt to isolate b or a power of b ***(M1)***

correct working ***(A1)***

$b = \frac{a}{a^3b^2}$ OR $b^3 = a^{-2}$ OR $b^{-1} = (ab)^2$ OR $b^3 = \frac{1}{a^2}$

$b = \frac{1}{a^2b^2}$ OR $b = (ab)^{-2}$ OR $3\ \log_{ab} b = -2\ \log_{ab} a$ OR $-\log_{ab} b = 2\ \log_{ab} ab$ ***A1***

$\log_{ab} b = -2$ ***AG***

METHOD 2

$(ab)^3 = a$ *(A1)*

taking logarithm to base ab on both sides *(M1)*

$\log_{ab}(ab)^3 = \log_{ab} a$ OR $\log_{ab} a^3 b^3 = \log_{ab} a$

correct application of log rules leading to equation in terms of $\log_{ab}$ *(A1)*

$3\log_{ab} a + 3\log_{ab} b = \log_{ab} a$ OR $3\log_{ab} b = -2\log_{ab} a$ OR $\log_{ab} b^3 = \log_{ab} a^{-2}$

$\log_{ab} b = \log_{ab} a^{-\frac{2}{3}}$ OR $\log_{ab} b = -\frac{2}{3}\log_{ab} a$ OR $\log_{ab} b = -\frac{2}{3}(3)$ *A1*

$\log_{ab} b = -2$ *AG*

Note: Candidates may substitute $\log_{ab} a = 3$ at any point in their working.

METHOD 3

$\log_{ab} a = 3$

writing in terms of base a *(M1)*

$\frac{\log_a a}{\log_a ab} (= 3)$

correct application of log rules *(A1)*

$\frac{\log_a a}{\log_a a + \log_a b} (= 3)$ OR $\frac{1}{1 + \log_a b} (= 3)$ OR $3\log_a b = -2$ OR $\log_a b = -\frac{2}{3}$

writing $\log_{ab} b$ in terms of base a *(A1)*

$\log_{ab} b = \frac{\log_a b}{\log_a a + \log_a b}$

correct working ***A1***

$\log_{ab} b = \frac{-\frac{2}{3}}{1-\frac{2}{3}}$ OR $\frac{\left(-\frac{2}{3}\right)}{\left(\frac{1}{3}\right)}$

$\log_{ab} b = -2$ ***AG***

METHOD 4

$\log_{ab} ab = 1$ ***A2***

$\log_{ab} a + \log_{ab} b = 1$ ***(A1)***

$3 + \log_{ab} b = 1$ ***A1***

$\log_{ab} b = -2$ ***AG***

[4 marks]

(b.ii) Hence find the value of $\log_{ab} \frac{\sqrt[3]{a}}{\sqrt{b}}$. [4]

Markscheme

applying the quotient rule or product rule for logs

$\log_{ab} \frac{\sqrt[3]{a}}{\sqrt{b}} = \log_{ab} \sqrt[3]{a} - \log_{ab} \sqrt{b}$ OR
$\log_{ab} \frac{\sqrt[3]{a}}{\sqrt{b}} = \log_{ab} \sqrt[3]{a} + \log_{ab} \frac{1}{\sqrt{b}}$ ***(A1)***

correct working ***(A1)***

$= \frac{1}{3}\log_{ab} a - \frac{1}{2}\log_{ab} b$ OR $\log_{ab} ab - \log_{ab} \sqrt{b}$

$= \frac{1}{3} \cdot 3 - \frac{1}{2}(-2)$ ***(A1)***

$= 2$ ***A1***

Note: Award ***A1A0A0A1*** for a correct answer with no working.

[4 marks]