Functions, exponents and logarithms (no GDC) [69 marks]

1. [Maximum mark: 5]

    It is given that $\log_{10} a = \frac{1}{3}$, where $a > 0$.

    Find the value of

    (a) $\log_{10} \left(\frac{1}{a} \right)$; [2]

    (b) $\log_{1000} a$. [3]

2. [Maximum mark: 5]

    Solve the equation $2 \ln x = \ln 9 + 4$. Give your answer in the form $x = pe^q$ where $p, q \in \mathbb{Z}^+$. [5]

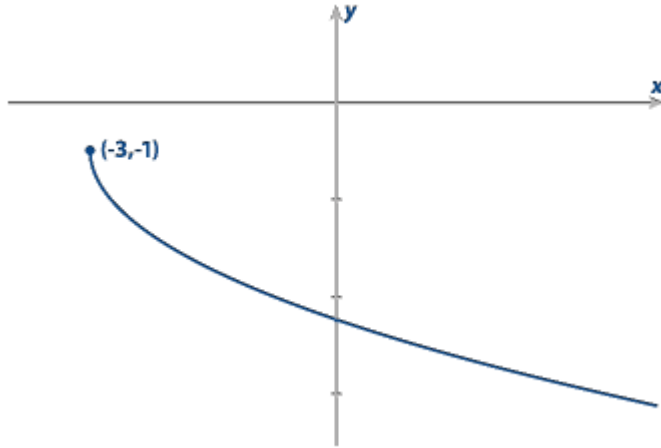3. [Maximum mark: 6]

    The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = x - 2$ and $g(x) = ax + b$, where $a, b \in \mathbb{R}$.

    Given that $(f \circ g)(2) = -3$ and $(g \circ f)(1) = 5$, find the value of a and the value of b. [6]

4. [Maximum mark: 14]

    The following diagram shows the graph of $y = -1 - \sqrt{x + 3}$ for $x \geq -3$.

(-3,-1)

(a)   Describe a sequence of transformations that transforms the
        graph of $y = \sqrt{x}$ for $x \geq 0$ to the graph of
        $y = -1 - \sqrt{x+3}$ for $x \geq -3$.   [3]

A function f is defined by $f(x) = -1 - \sqrt{x+3}$ for $x \geq -3$.

(b)   State the range of f.   [1]

(c)   Find an expression for $f^{-1}(x)$, stating its domain.   [5]

(d)   Find the coordinates of the point(s) where the graphs of
        $y = f(x)$ and $y = f^{-1}(x)$ intersect.   [5]

5.   [Maximum mark: 5]   24M.1.AHL.TZ2.2
        Solve $3 \times 9^x + 5 \times 3^x - 2 = 0$.   [5]

6.   [Maximum mark: 7]   23M.1.SL.TZ1.2
        The function f is defined by $f(x) = \frac{7x+7}{2x-4}$ for $x \in \mathbb{R}, x \neq 2$.

(a)   Find the zero of $f(x)$.   [2]

(b)   For the graph of $y = f(x)$, write down the equation of

(b.i)   the vertical asymptote;   [1]

(b.ii)   the horizontal asymptote.   [1]

(c)   Find $f^{-1}(x)$, the inverse function of $f(x)$.   [3]

7.   [Maximum mark: 7]
The functions f and g are defined for $x \in \mathbb{R}$ by

$$f(x) = ax + b, \text{ where } a, b \in \mathbb{Z}$$

$$g(x) = x^2 + x + 3.$$

Find the two possible functions f such that
$(g \circ f)(x) = 4x^2 - 14x + 15.$

[7]

8.   [Maximum mark: 5]
A function f is defined by $f(x) = 1 - \frac{1}{x-2}$, where $x \in \mathbb{R}, x \neq 2$.

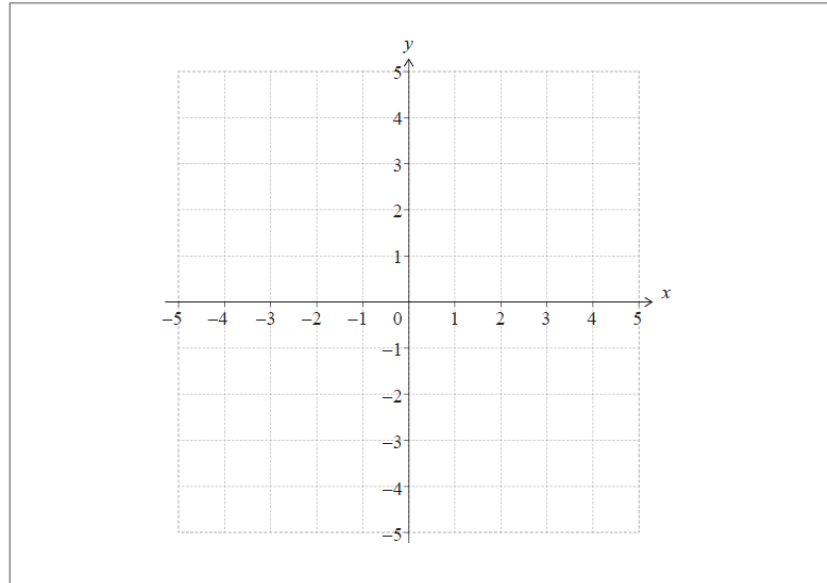(a)   The graph of $y = f(x)$ has a vertical asymptote and a
        horizontal asymptote.

        Write down the equation of

(a.i)   the vertical asymptote;   [1]

(a.ii)   the horizontal asymptote.   [1]

(b)   Find the coordinates of the point where the graph of
        $y = f(x)$ intersects

(b.i)   the y-axis;   [1]

(b.ii)   the x-axis.   [1]

(c)   On the following set of axes, sketch the graph of $y = f(x)$,
       showing all the features found in parts (a) and (b).

[1]

9.   [Maximum mark: 15]   22N.1.SL.TZ0.8
      Calculate the value of each of the following logarithms:

(a.i) $\log_2 \frac{1}{16}$.   [2]

(a.ii) $\log_9 3$.   [2]

(a.iii) $\log_{\sqrt{3}} 81$.   [3]

It is given that $\log_{ab} a = 3$, where $a,\ b \in \mathbb{R}^+,\ ab \neq 1$.

(b.i)   Show that $\log_{ab} b = -2$.   [4]

(b.ii)   Hence find the value of $\log_{ab} \frac{\sqrt[3]{a}}{\sqrt{b}}$.   [4]