Paper 3 practice [80 marks]

1. [Maximum mark: 25] SPM.3.AHL.TZ0.2

This question asks you to investigate some properties of the sequence of functions of the form $f_n(x) = \cos(n \arccos x)$, $-1 \le x \le 1$ and $n \in \mathbb{Z}^+$.

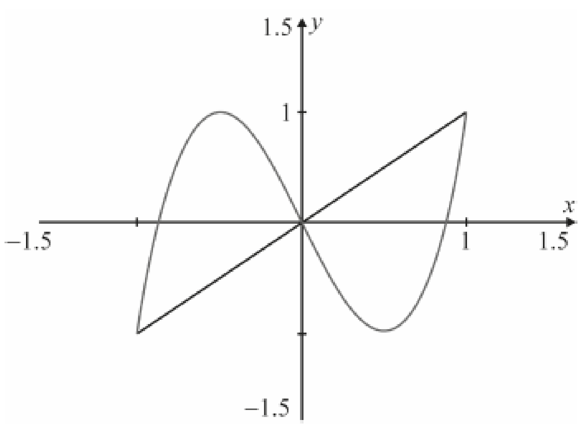
Important: When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

(a) On the same set of axes, sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for $-1 \le x \le 1$. [2]

Markscheme

correct graph of $y = f_1(x)$ ***A1***

correct graph of $y = f_3(x)$ ***A1***



[2 marks]

For odd values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for odd values of n describing, in terms of n, the number of

(b.i) local maximum points; [3]

Markscheme

graphical or tabular evidence that n has been systematically varied ***M1***

eg $n = 3$, 1 local maximum point and 1 local minimum point

$n = 5$, 2 local maximum points and 2 local minimum points

$n = 7$, 3 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-1}{2}$ local maximum points ***A1***

[3 marks]

(b.ii) local minimum points; [1]

Markscheme

$\frac{n-1}{2}$ local minimum points ***A1***

Note: Allow follow through from an incorrect local maximum formula expression.

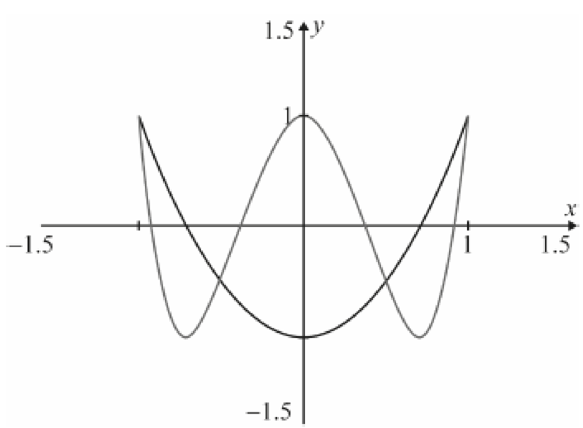
[1 mark]

(c) On a new set of axes, sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for $-1 \le x \le 1$. [2]

Markscheme

correct graph of $y = f_2(x)$ ***A1***

correct graph of $y = f_4(x)$ ***A1***



[2 marks]

For even values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for even values of $n$describing, in terms of n, the number of (d.i) local maximum points; [3]

Markscheme

graphical or tabular evidence that n has been systematically varied ***M1***

eg n = 2, 0 local maximum point and 1 local minimum point

n = 4, 1 local maximum points and 2 local minimum points

n = 6, 2 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-2}{2}$ local maximum points ***A1***

[3 marks]

(d.ii) local minimum points. [1]

Markscheme

$\frac{n}{2}$ local minimum points ***A1***

[1 mark]

(e) Solve the equation $f_n{}'(x) = 0$ and hence show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos\frac{k\pi}{n}$ where $k \in \mathbb{Z}^+$ and 0 < k < n. [4]

Markscheme

$f_n(x) = \cos\left(n \arccos\left(x\right)\right)$

$f_n{}'(x) = \frac{n \sin(n \arccos(x))}{\sqrt{1-x^2}}$ ***M1A1***

Note: Award ***M1*** for attempting to use the chain rule.

$f_n{}'(x) = 0 \Rightarrow n \sin\left(n \arccos\left(x\right)\right) = 0$ ***M1***

$n \arccos\left(x\right) = k\pi \quad \left(k \in \mathbb{Z}^+\right)$ ***A1***

leading to

$x = \cos\frac{k\pi}{n}$ ($k \in \mathbb{Z}^+$ and 0 < k < n) ***AG***

[4 marks]

The sequence of functions, $f_n(x)$, defined above can be expressed as a sequence of polynomials of degree n.

(f) Use an appropriate trigonometric identity to show that $f_2(x) = 2x^2 - 1$. [2]

Markscheme

$f_2(x) = \cos(2 \arccos x)$

$= 2(\cos(\arccos x))^2 - 1$ ***M1***

stating that $(\cos(\arccos x)) = x$ ***A1***

so $f_2(x) = 2x^2 - 1$ ***AG***

[2 marks]

Consider $f_{n+1}(x) = \cos((n+1) \arccos x)$.

(g) Use an appropriate trigonometric identity to show that $f_{n+1}(x) = \cos(n \arccos x)\cos(\arccos x) - \sin(n \arccos x)\sin(\arccos x)$. [2]

Markscheme

$f_{n+1}(x) = \cos((n+1) \arccos x)$

$= \cos(n \arccos x + \arccos x)$ ***A1***

use of $\cos(A + B) = \cos A \cos B - \sin A \sin B$ leading to ***M1***

$= \cos(n \arccos x)\cos(\arccos x) - \sin(n \arccos x)\sin(\arccos x)$ ***AG***

[2 marks]

(h.i) Hence show that $f_{n+1}(x) + f_{n-1}(x) = 2xf_n(x), n \in \mathbb{Z}^+$. [3]

Markscheme

$f_{n-1}(x) = \cos((n-1) \arccos x)$ ***A1***

$= \cos(n \arccos x)\cos(\arccos x) + \sin(n \arccos x)\sin(\arccos x)$ ***M1***

$f_{n+1}(x) + f_{n-1}(x) = 2\cos(n \arccos x)\cos(\arccos x)$ ***A1***

$= 2x f_n(x)$ ***AG***

[3 marks]

(h.ii) Hence express $f_3(x)$ as a cubic polynomial. [2]

Markscheme

$f_3(x) = 2x f_2(x) - f_1(x)$ ***(M1)***

$= 2x(2x^2 - 1) - x$

$= 4x^3 - 3x$ ***A1***

[2 marks]

2. [Maximum mark: 29] EXM.3.AHL.TZ0.3

This question will investigate methods for finding definite integrals of powers of trigonometrical functions.

Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx, \; n \in \mathbb{N}.$

(a) Find the exact values of I_0, I_1 and I_2. [6]

Markscheme

$I_0 = \int_0^{\frac{\pi}{2}} 1 \, dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$ ***M1A1***

$I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$ ***M1A1***

$I_2 = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} \, dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$ ***M1A1***

[6 marks]

(b.i) Use integration by parts to show that $I_n = \frac{n-1}{n} I_{n-2},\ n \geqslant 2$. [5]

Markscheme

$u = \sin^{n-1} x \qquad v = -\cos x$

$\frac{du}{dx} = (n-1)\sin^{n-2} x \cos x \quad \frac{dv}{dx} = \sin x$

$I_n = \left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x \cos^2 x \, dx$ ***M1A1A1***

$= 0 + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x \left(1 - \sin^2 x\right) dx = (n-1)\left(I_{n-2} - I_n\right)$ ***M1A1***

$\Rightarrow nI_n = (n-1)I_{n-2} \Rightarrow I_n = \frac{(n-1)}{n} I_{n-2}$ ***AG***

[6 marks]

(b.ii) Explain where the condition $n \geqslant 2$ was used in your proof. [1]

Markscheme

need $n \geqslant 2$ so that $\sin^{n-1} \frac{\pi}{2} = 0$ in $\left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}}$ ***R1***

[1 mark]

(c) Hence, find the exact values of I_3 and I_4. [2]

Markscheme

$I_3 = \frac{2}{3} I_1 = \frac{2}{3} \qquad I_4 = \frac{3}{4} I_2 = \frac{3\pi}{16}$ ***A1A1***

[2 marks]

Let $J_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx,\ n \in \mathbb{N}$.

(d) Use the substitution $x = \frac{\pi}{2} - u$ to show that $J_n = I_n$. [4]

Markscheme

$x = \frac{\pi}{2} - u \Rightarrow \frac{dx}{du} = -1$ ***A1***

$$J_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \int_{\frac{\pi}{2}}^{0} -\cos^n\left(\frac{\pi}{2} - u\right)du = -\int_{\frac{\pi}{2}}^{0} \sin^n u\,du = \int_0^{\frac{\pi}{2}} \sin^n u\,du = I_n$$

M1A1A1AG

[4 marks]

(e) Hence, find the exact values of J_5 and J_6 [2]

Markscheme

$J_5 = I_5 = \frac{4}{5}I_3 = \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$ $J_6 = I_6 = \frac{5}{6}I_4 = \frac{5}{6} \times \frac{3\pi}{16} = \frac{5\pi}{32}$ ***A1A1***

[2 marks]

Let $T_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx,\ n \in \mathbb{N}.$

(f) Find the exact values of T_0 and T_1. [3]

Markscheme

$$T_0 = \int_0^{\frac{\pi}{4}} 1\,dx = [x]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$$ ***A1***

$$T_1 = \int_0^{\frac{\pi}{4}} \tan dx = [-\ln|\cos x|]_0^{\frac{\pi}{4}} = -\ln\frac{1}{\sqrt{2}} = \ln\sqrt{2}$$ ***M1A1***

[3 marks]

(g.i) Use the fact that $\tan^2 x = \sec^2 x - 1$ to show that $T_n = \frac{1}{n-1} - T_{n-2},\ n \geqslant 2$

. [3]

Markscheme

$$T_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x \, dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \left(\sec^2 x - 1\right) dx$$

M1

$$\int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - T_{n-2} = \frac{1}{n-1} - T_{n-2}$$

A1A1AG

[3 marks]

(g.ii) Explain where the condition $n \geqslant 2$ was used in your proof. [1]

Markscheme

need $n \geqslant 2$ so that the powers of tan in $\int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$ are not negative ***R1***

[1 mark]

(h) Hence, find the exact values of T_2 and T_3. [2]

Markscheme

$T_2 = 1 - T_0 = 1 - \frac{\pi}{4}$ ***A1***

$T_3 = \frac{1}{2} - T_1 = \frac{1}{2} - \ln\sqrt{2}$ ***A1***

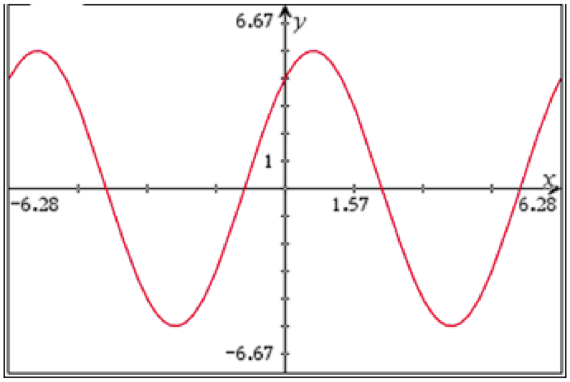
[2 marks]

3. [Maximum mark: 26] EXM.3.AHL.TZ0.5

This question investigates the sum of sine and cosine functions

(a.i) Sketch the graph $y = 3\sin x + 4\cos x$, for $-2\pi \leqslant x \leqslant 2\pi$ [1]

Markscheme



A1

[1 mark]

(a.ii) Write down the amplitude of this graph [1]

Markscheme

5 ***A1***

[1 mark]

(a.iii) Write down the period of this graph [1]

Markscheme

2π ***A1***

[1 mark]

The expression $3\sin x + 4\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $A,\ B \in \mathbb{R}^+$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

(b.i) Use your answers from part (a) to write down the value of A, B and D. [1]

Markscheme

$A = 5, B = 1, D = 0$ ***A1***

[1 mark]

(b.ii) Find the value of C. [2]

Markscheme

maximum at $x = 0.644$ ***M1***

So $C = -0.644$ ***A1***

[2 marks]

(c.i) Find $\arctan\frac{3}{4}$, giving the answer to 3 significant figures. [1]

Markscheme

0.644 ***A1***

[1 mark]

(c.ii) Comment on your answer to part (c)(i). [1]

Markscheme

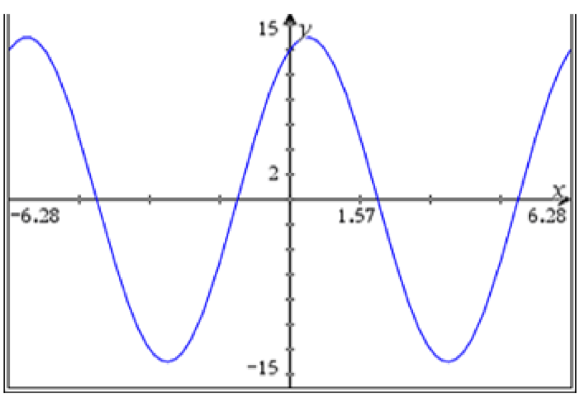
it appears that $C = -\arctan\frac{3}{4}$ ***A1***

[1 mark]

The expression $5\sin x + 12\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $A,\ B \in \mathbb{R}^+$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

(d) By considering the graph of $y = 5\sin x + 12\cos x$, find the value of A, B, C and D. [5]

Markscheme



M1

$A = 13$ ***A1***

$B = 1$ and $D = 0$ ***A1***

maximum at $x = 0.395$ ***M1***

So C = −0.395 $\left(= -\arctan\frac{5}{12}\right)$ ***A1***

[5 marks]

In general, the expression $a\sin x + b\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $a,\ b,\ A,\ B \in \mathbb{R}^+$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

Conjecture an expression, in terms of a and b, for

(e.i) A. [1]

Markscheme

$A = \sqrt{a^2 + b^2}$ ***A1***

[1 mark]

(e.ii) B. [1]

Markscheme

$B = 1$ ***A1***

[1 mark]

(e.iii) C. [1]

Markscheme

$C = -\arctan\frac{a}{b}$ ***A1***

[1 mark]

(e.iv) D. [1]

Markscheme

$D = 0$ ***A1***

[1 mark]

The expression $a \sin x + b \cos x$ can also be written in the form $\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right)$.

Let $\frac{a}{\sqrt{a^2+b^2}} = \sin\theta$

(f.i) Show that $\frac{b}{\sqrt{a^2+b^2}} = \cos\theta$. [2]

Markscheme

EITHER

use of a right triangle and Pythgoras' to show the missing side length is b ***M1A1***

OR

Use of $\sin^2\theta + \cos^2\theta = 1$, leading to the required result ***M1A1***

[2 marks]

(f.ii) Show that $\frac{a}{b} = \tan\theta$. [1]

Markscheme

EITHER

use of a right triangle, leading to the required result. ***M1***

OR

Use of $\tan\theta = \frac{\sin\theta}{\cos\theta}$, leading to the required result. ***M1***

[1 mark]

(g) Hence prove your conjectures in part (e). [6]

Markscheme

$a \sin x + b \cos x = \sqrt{a^2+b^2}\left(\sin\theta \sin x + \cos\theta \cos x\right)$ ***M1***

$a \sin x + b \cos x = \sqrt{a^2 + b^2}\,(\cos(x - \theta))$ ***M1A1***

So $A = \sqrt{a^2 + b^2}$, $B = 1$ and $D = 0$ ***A1***

And $C = -\theta$ ***M1***

So $C = -\arctan\frac{a}{b}$ ***A1***

[6 marks]