Paper 3 practice [80 marks]

1. [Maximum mark: 25] SPM.3.AHL.TZ0.2

This question asks you to investigate some properties of the sequence of functions of the form $f_n(x) = \cos(n \arccos x)$, −1 ≤ x ≤ 1 and $n \in \mathbb{Z}^+$.

Important: When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

(a) On the same set of axes, sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for −1 ≤ x ≤ 1. [2]

For odd values of n > 2, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for odd values of n describing, in terms of n, the number of

(b.i) local maximum points; [3]

(b.ii) local minimum points; [1]

(c) On a new set of axes, sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for −1 ≤ x ≤ 1. [2]

For even values of n > 2, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for even values of $n$describing, in terms of n, the number of

(d.i) local maximum points; [3]

(d.ii) local minimum points. [1]

(e) Solve the equation $f_n{}'(x) = 0$ and hence show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos\frac{k\pi}{n}$ where $k \in \mathbb{Z}^+$ and 0 < k < n. [4]

The sequence of functions, $f_n(x)$, defined above can be expressed as a sequence of polynomials of degree n.

(f) Use an appropriate trigonometric identity to show that $f_2(x) = 2x^2 - 1$. [2]

Consider $f_{n+1}(x) = \cos((n+1)\arccos x)$.

(g) Use an appropriate trigonometric identity to show that $f_{n+1}(x) = \cos(n \arccos x)\cos(\arccos x) - \sin(n \arccos x)\sin(\arccos x)$. [2]

(h.i) Hence show that $f_{n+1}(x) + f_{n-1}(x) = 2xf_n(x), n \in \mathbb{Z}^+$.

[3]

(h.ii) Hence express $f_3(x)$ as a cubic polynomial. [2]

2. [Maximum mark: 29] EXM.3.AHL.TZ0.3

This question will investigate methods for finding definite integrals of powers of trigonometrical functions.

Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx, \; n \in \mathbb{N}$.

(a) Find the exact values of I_0, I_1 and I_2. [6]

(b.i) Use integration by parts to show that $I_n = \frac{n-1}{n} I_{n-2}, \; n \geqslant 2$. [5]

(b.ii) Explain where the condition $n \geqslant 2$ was used in your proof. [1]

(c) Hence, find the exact values of I_3 and I_4. [2]

Let $J_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx, \; n \in \mathbb{N}$.

(d) Use the substitution $x = \frac{\pi}{2} - u$ to show that $J_n = I_n$. [4]

(e) Hence, find the exact values of J_5 and J_6 [2]

Let $T_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx, \; n \in \mathbb{N}$.

(f) Find the exact values of T_0 and T_1. [3]

(g.i) Use the fact that $\tan^2 x = \sec^2 x - 1$ to show that $T_n = \frac{1}{n-1} - T_{n-2}, \; n \geqslant 2$
. [3]

(g.ii) Explain where the condition $n \geqslant 2$ was used in your proof. [1]

(h) Hence, find the exact values of T_2 and T_3. [2]

3. [Maximum mark: 26] EXM.3.AHL.TZ0.5

This question investigates the sum of sine and cosine functions

(a.i) Sketch the graph $y = 3\sin x + 4\cos x$, for $-2\pi \leqslant x \leqslant 2\pi$ [1]

(a.ii) Write down the amplitude of this graph [1]

(a.iii) Write down the period of this graph [1]

The expression $3\sin x + 4\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $A,\ B \in \mathbb{R}^+$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

(b.i) Use your answers from part (a) to write down the value of A, B and D. [1]

(b.ii) Find the value of C. [2]

(c.i) Find $\arctan\frac{3}{4}$, giving the answer to 3 significant figures. [1]

(c.ii) Comment on your answer to part (c)(i). [1]

The expression $5\sin x + 12\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $A,\ B \in \mathbb{R}^+$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

(d) By considering the graph of $y = 5\sin x + 12\cos x$, find the value of A, B, C and D. [5]

In general, the expression $a\sin x + b\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $a,\ b,\ A,\ B \in \mathbb{R}^+$ and $C,\ D \in \mathbb{R}$ and $-\pi < C \leqslant \pi$.

Conjecture an expression, in terms of a and b, for

(e.i) A. [1]

(e.ii) B. [1]

(e.iii) C. [1]

(e.iv) D. [1]

The expression $a\sin x + b\cos x$ can also be written in the form

$\sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right)$.

Let $\frac{a}{\sqrt{a^2+b^2}} = \sin\theta$

(f.i) Show that $\frac{b}{\sqrt{a^2+b^2}} = \cos\theta$. [2]

(f.ii) Show that $\frac{a}{b} = \tan\theta$. [1]

(g) Hence prove your conjectures in part (e). [6]