Mathematics PreDP1

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Chapter 1

Algebra

1.1 Algebraic manipulation

EXPANSION AND FACTORIZATION

Consider the expression $5(x + y)$ we can **expand** this expression by using the distributive property of multiplication over addition to get $5x + 5y$. On the other hand, if we started with the expression $3a + 6b$ we could **factor** this into $3(a+2b)$. In this section we will practice expanding an factoring algebraic expressions.

Exercise 2.1.1a
Expand the following expressions:

(a) $7(x + 3y)$ (b) $a(5 - b)$ (c) $\sqrt{2}(x + 2\sqrt{2})$

(d) $\frac{1}{2}(8 + 6c)$ (e) $\frac{x}{y}(2x - 3y)$ (f) $\frac{3}{b}(5b + 4b^2)$

(g) $xy(x + y - z)$ (h) $\frac{5}{ab}(a - b + a^2b)$ (i) $\frac{2}{pqr}(pq - pr - qr)$

Exercise 2.1.1b
Factorize the following expressions:

(a) $5x + 10y$ (b) $6a + 12b$ (c) $10x^2 + 5x$

(d) $15xy + 25y^2$ (e) $24x^2y + 16xy^2$ (f) $18a^3b^2 + 12a^2b^3$

(g) $30m^4n^2 + 45m^3n^3$ (h) $8x^3 - 12x^2 + 4x$ (i) $50x^2y^3 - 25x^3y^2 + 75x^4y$

(j) $9x^2yz + 18xy^2z + 27xyz^2$ (k) $15x^3y^2z + 25x^2yz^2 - 10xyz^3$ (l) $40a^2b^3c - 20a^3bc^2 + 8a^2bc^3$

MULTIPLYING BINOMIALS

A binomial is an expression which is a sum (or a difference) of two terms (called monomials). The following are examples of binomials: $3x + 5,\ xyz + a^5,\ \sqrt{5} + x^4$. When multiplying binomials we use the distributive property of multiplication over addition:

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$$

Note that the distributive property was applied twice in the example above.

In the following examples and exercises you will practice multiplying binomials.

Worked example 2.1.2

Expand and simplify

$$(\sqrt{2} + 3)(5 - 3\sqrt{2})$$

$$\begin{aligned}&(\sqrt{2} + 3)(5 - 3\sqrt{2}) = \\ &= 5\sqrt{2} - 6 + 15 - 9\sqrt{2} = \\ &= 9 - 4\sqrt{2}\end{aligned}$$

Exercise 2.1.2a

Expand and simplify:

(a) $(3 + \sqrt{2})(2 - 2\sqrt{2})$

(b) $(5 + \sqrt{3})(3 + 2\sqrt{3})$

(c) $(2 - 3\sqrt{5})(5 - 2\sqrt{5})$

(d) $(4 + 2\sqrt{7})(1 - \sqrt{7})$

(e) $(\sqrt{2} + \sqrt{3})(\sqrt{2} + 2\sqrt{3})$

(f) $(2\sqrt{5} - 3\sqrt{2})(\sqrt{5} + \sqrt{2})$

(g) $(\sqrt{3} - 2\sqrt{7})(2\sqrt{3} + \sqrt{7})$

(h) $(\sqrt{5} + 2\sqrt{3})(\sqrt{5} + 3\sqrt{3})$

(i) $(\sqrt{7} - 3\sqrt{2})(\sqrt{7} - 2\sqrt{2})$

(j) $(a - b\sqrt{2})(b + c\sqrt{2})$

(k) $(x + y\sqrt{3})(z + x\sqrt{3})$

(l) $(p - q\sqrt{2})(q - p\sqrt{2})$

(m) $(2 - p\sqrt{q})(p + 3\sqrt{q})$

(n) $(a + b\sqrt{c})(b + a\sqrt{c})$

(o) $(\sqrt{a} - \sqrt{b})(2\sqrt{a} - 3\sqrt{b})$

(p) $(ab - c)(a + bc)$

(q) $(pq - q)(p + q)$

(r) $(xy - z)(3xy + 4z)$

There are some very important examples binomial multiplication that you should be familiar with:

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

$$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$$

$$(a+b)(a-b) = a^2 - b^2$$

Exercise 2.1.2b
Expand and simplify:

(a) $(1+\sqrt{2})^2$ (b) $(3+\sqrt{2})^2$ (c) $(5+2\sqrt{5})^2$

(d) $(1-\sqrt{7})^2$ (e) $(\sqrt{2}-\sqrt{3})^2$ (f) $(2\sqrt{5}-\sqrt{2})^2$

(g) $(3+\sqrt{7})(3-\sqrt{7})$ (h) $(5+2\sqrt{3})(5-2\sqrt{3})$ (i) $(\sqrt{7}-\sqrt{2})(\sqrt{7}+\sqrt{2})$

(j) $(x+2y)^2$ (k) $(3p+q\sqrt{2})^2$ (l) $(3d+2c\sqrt{3})^2$

(m) $(4x-3y)^2$ (n) $(2m-n\sqrt{5})^2$ (o) $(2c-d\sqrt{2})^2$

(p) $(x+5y)(x-5y)$ (q) $(3a+b\sqrt{3})(3a-b\sqrt{3})$ (r) $(r\sqrt{5}-s\sqrt{3})(r\sqrt{5}+s\sqrt{3})$

(s) $(2+3\sqrt{5})^2$ (t) $(4-\sqrt{3})^2$ (u) $(6+2\sqrt{2})^2$

(v) $(5+\sqrt{6})(5-\sqrt{6})$ (w) $(7+\sqrt{10})(7-\sqrt{10})$ (x) $(\sqrt{11}-\sqrt{3})(\sqrt{11}+\sqrt{3})$

(y) $(2x+3y)^2$ (z) $(5p-2q\sqrt{7})^2$ (zz) $(4d+c\sqrt{6})^2$

CHANGING THE SUBJECT OF A FORMULA

In some cases we are given a formula that involves several variables and we may want to rearrange it to make a different variable the subject of it. Consider:

$$x = \frac{y}{z}$$

This is a formula for x in terms of y and z. If we multiply both sides of this formula by z we get:

$$y = xz$$

and we have a formula for y in terms of x and z. We can then divide both sides by x to get:

$$z = \frac{y}{x}$$

a formula for z in terms of x and y. Note that when me make a variable the subject of a formula, then we require this variable to only appear on one side of the equation. We cannot have a formula for x in terms of x. Note that rearranging the formula often involves factoring, squaring, taking roots etc.

Worked example 2.1.3

Rearrange the following formula to make x its subject:

(a) $y = 3\sqrt{2x-7}$

(b) $y = \dfrac{3x+2}{ax+b}$

(a) Divide both sides by 3 and then square both sides to get:

$$\frac{y^2}{9} = 2x - 7$$

Now we add 7 to both sides and then divide by 2:

$$\frac{\frac{y^2}{9}+7}{2} = x$$

Finally we can multiply the denominator and numerator of the fraction by 9 to get:

$$x = \frac{y^2+63}{18}$$

(b)We first multiply both sides by the denominator of the left hand side $(ax+b)$:

$$axy + by = 3x + 2$$

Now we move all terms with x to one side and all other terms to the other side:

$$axy - 3x = 2 - by$$

We can now factor out x on the left hand side:

$$x(ay-3) = 2 - by$$

And finally we divide by the bracket $(ay-3)$:

$$x = \frac{2-by}{ay-3}$$

Exercise 2.1.3

Rearrange the following formulae to make x their subject:

(a) $y = 2x + 1$

(b) $y = \dfrac{x+3}{2}$

(c) $y = \dfrac{2x-5}{3}$

(d) $y = ax + b$

(e) $y = \dfrac{x-c}{d}$

(f) $y = \dfrac{px+q}{r}$

(g) $a = 2x + bx$

(h) $z = cx - dx$

(i) $w = \dfrac{x}{2} + ax$

(j) $c = \sqrt{x-1}$

(k) $b = a\sqrt{bx+d}$

(l) $y = \sqrt[3]{3x - ax}$

(m) $a = \frac{1}{\sqrt{2x+5}}$

(n) $d = c\sqrt{ex - f}$

(o) $z = \sqrt[4]{5x + \sqrt{2}x}$

(p) $m = \dfrac{a}{\sqrt{x+b}}$

(q) $v = \dfrac{3}{\sqrt{ax-t}}$

(r) $k = \dfrac{3x+b}{x-7}$

(s) $w = \dfrac{p}{x} + q$

(t) $y = \dfrac{ax+b}{cx+d}$

(u) $c = \dfrac{2x-3}{4x+5}$

(v) $y = x^2 - 3$

(w) $a = bx^2$

(x) $p = \dfrac{x^2}{q} + r$

Pascal's triangle and binomial expansion

The following triangle is known as the Pascal's triangle.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

The triangle consists of 1s on the left and right sides and each other number is the sum of the number above and to the right with the number above and to the left. The rows of the triangle are counted starting from zero, so for example 1 2 1 is the second row.

1. Write down two more rows of the triangle.

2. Expand the following and write your answer in descending powers of a:

$$(a+b)^0 \qquad (a+b)^1 \qquad (a+b)^2$$

3. By writing $(a+b)^3$ as $(a+b)^2(a+b)$ expand it and write in descending powers of a.

4. What do you notice about coefficients of terms in your expansions in parts 2. and 3.?

5. What do you notice about powers of a and b in your expansions in parts 2. and 3.?

6. Conjecture the expansion of $(a+b)^4$ and test your conjecture by writing $(a+b)^4$ as $(a+b)^3(a+b)$.

7. Expand $(a+b)^7$.

8. Expand and simplify the following

$$(1+\sqrt{2})^4 \qquad (2+\sqrt{3})^6 \qquad (3-\sqrt{5})^5$$

SHORT TEST

1. [4 *points*]

Expand and simplify (if possible):

(a) $\frac{x}{y}(3xy - y)$

(b) $(2 + \sqrt{2})(1 - 3\sqrt{2})$

(c) $(a - 3b)^2$

(d) $(y - \sqrt{5})(y + \sqrt{5})$

2. [3 *points*]

Factorize the following expressions:

(a) $6x^2 + 9xy$

(b) $2a^2b + 4b^2a - 6ab$

(c) $12x^3 - 8x^2 + 16x$

3. [3 *points*]

Make z the subject of the following formulae:

(a) $b = \dfrac{2z}{c}$

(b) $x = \dfrac{y}{\sqrt{z - 2}}$

(c) $y = \dfrac{z - a}{z + a}$

SHORT TEST

SOLUTIONS

1. [4 *points*]

Expand and simplify (if possible):

(a) $\frac{x}{y}(3xy - y)$

(a) $= 3x^2 - x$

(b) $(2 + \sqrt{2})(1 - 3\sqrt{2})$

(b) $= 2 - 6\sqrt{2} + \sqrt{2} - 6 = -4 - 5\sqrt{2}$

(c) $(a - 3b)^2$

(c) $= a^2 - 6ab + 9b^2$

(d) $(y - \sqrt{5})(y + \sqrt{5})$

(d) $= y^2 - 5$

2. [3 *points*]

Factorize the following expressions:

(a) $6x^2 + 9xy$

(a) $= 3x(2x + 3y)$

(b) $2a^2b + 4b^2a - 6ab$

(b) $= 2ab(a + 2b - 3)$

(c) $12x^3 - 8x^2 + 16x$

(c) $= 4x(3x^2 - 2x + 4)$

3. [3 *points*]

Make z the subject of the following formulae:

(a) $b = \dfrac{2z}{c} \quad \Rightarrow \quad bc = 2z \quad \Rightarrow \quad z = \dfrac{bc}{2}$

(b) $x = \dfrac{y}{\sqrt{z-2}} \quad \Rightarrow \quad \sqrt{z-2} = \dfrac{y}{x} \quad \Rightarrow \quad z - 2 = \dfrac{y^2}{x^2} \quad \Rightarrow \quad z = \dfrac{y^2}{x^2} + 2$

(c) $y = \dfrac{z-a}{z+a} \quad \Rightarrow \quad yz + ay = z - a \Rightarrow ay + a = z - yz \Rightarrow ay + a = z(1-y) \Rightarrow z = \dfrac{ay + a}{1 - y}$

1.2 Further manipulation

FACTORIZATION

Consider the expression:

$$x^2 - 16$$

It can be viewed as a difference of squares x^2 and 4^2 and then factorized using the difference of squares formula:

$$x^2 - 16 = (x - 4)(x + 4)$$

Worked example 2.2.1

Factorize $x^4 - 100$

We start by noting that we have:

$$x^4 - 100 = (x^2)^2 - 10^2$$

So we can apply the difference of squares formula to get:

$$(x^2 - 10)(x^2 + 10)$$

Now note that the first bracket can be written as

$$x^2 - (\sqrt{10})^2$$

so it can be further factored to:

$$(x - \sqrt{10})(x + \sqrt{10})(x^2 + 10)$$

Exercise 2.2.1

Factorize the following expressions using difference of squares formula:

(a) $x^2 - 9$

(b) $a^2 - 1$

(c) $p^2 - 3$

(d) $a^2 - 4b^2$

(e) $9x^2 - 25y^2$

(f) $c^2 - 2d^2$

(g) $100x^2 - 2$

(h) $25 - 4k^2$

(i) $8 - 3q^2$

(j) $p^4 - 1$

(k) $z^4 - 16$

(l) $t^8 - 1$

Recall that we have:

$$(x+3)^2 = x^2 + 6x + 9$$

Which means that the expression $x^2 + 6x + 9$ can be factorize into $(x+3)^2$. It is very useful to be able to recognize the expansion of $(a+b)^2$ and $(a-b)^2$.

Exercise 2.2.2 Write each expression as a square of sum or difference:

(a) $x^2 + 4x + 4$ (b) $x^2 - 10x + 25$ (c) $x^2 - 8x + 16$

(d) $9x^2 - 6x + 1$ (e) $4x^2 - 4x + 1$ (f) $81x^2 + 18x + 1$

(g) $4x^2 + 12x + 9$ (h) $9x^2 + 30x + 25$ (i) $25x^2 - 40x + 16$

Recognizing the formula for square of sum or difference together with formula for difference of squares can be used to factorize quadratic trinomials.

Worked example 2.2.3

Factorize $x^2 + 4x + 3$

We rewrite the expression to use the formula for square of sum:

$$x^2 + 4x + 3 = x^2 + 4x + 4 - 1 = (x+2)^2 - 1$$

Now we can apply the formula for difference of squares:

$$(x+2)^2 - 1 = (x+2-1)(x+2+1) = (x+1)(x+3)$$

Exercise 2.2.3
Factorize the following expressions:

(a) $x^2 + 6x + 8$ (b) $x^2 + 2x - 8$ (c) $x^2 + 10x + 24$

(d) $4x^2 - 4x - 3$ (e) $4x^2 + 4x - 15$ (f) $9x^2 - 6x - 3$

(g) $x^2 + 2x - 1$ (h) $x^2 - 4x + 1$ (i) $9x^2 + 6x - 1$

(j) $4x^2 + 12x + 5$ (k) $9x^2 - 12x - 5$ (l) $16x^2 + 8x - 15$

We will look at a different approach at factoring trinomials by considering the expansion of $(x+2)(x+5)$:

$$(x+2)(x+5) = x^2 + 5x + 2x + 10 = x^2 + 7x + 10$$

In the final form of the expression the coefficient of x is 7 and it comes from adding 2 and 5, the constant term is 10 and it comes from multiplying 2 and 5. So if we were to factorize $x^2 + 7x + 10$ we would need to find two numbers that added together give 7 and multiplied together give 10.

Worked example 2.2.4

Factorize $x^2 + x - 12$

We are looking for two numbers that added together give 1 and multiplied together give -12. Note that this means one of the numbers will be positive and the other negative. We should look at the integers that multiplied produce -12:

factors	sum
1 and -12	-11
12 and -1	11
2 and -6	-4
6 and -2	4
3 and -4	-1
4 and -3	1

We found the numbers satisfying the conditions (4 and -3), so we get:

$$x^2 + x - 12 = (x+4)(x-3)$$

Exercise 2.2.4
Factorize the following expressions:

(a) $x^2 + 6x + 5$ (b) $x^2 + 6x + 8$ (c) $x^2 + 8x + 15$

(d) $x^2 - 7x + 12$ (e) $x^2 - 7x + 10$ (f) $x^2 - 4x + 3$

(g) $x^2 + 2x - 15$ (h) $x^2 - 3x - 18$ (i) $x^2 + 5x - 6$

(j) $x^2 + x - 30$ (k) $x^2 - 2x - 24$ (l) $x^2 + 4x - 21$

(m) $x^2 + 10x + 16$ (n) $x^2 - 9x - 22$ (o) $x^2 + 7x - 30$

(p) $x^2 + (a+b)x + ab$ (q) $x^2 + (p-3q)x - 3pq$ (r) $x^2 - (r-s)x - rs$

Another approach to factoring trinomials like $x^2 + 7x + 10$ is to **split the middle** term so that the ratio of the coefficients of the first and second term is the same as the ratio of coefficients of the third and fourth term:

$$x^2 + 7x + 10 = x^2 + 2x + 5x + 10$$

Note that the ratios are equal $\frac{1}{2} = \frac{5}{10}$, now we can factorize the terms in pairs:

$$x^2 + 2x + 5x + 10 = x(x + 2) + 5(x + 2) = (x + 2)(x + 5)$$

Worked example 2.2.5

Factorize $x^2 - 9x + 18$ by splitting the middle term.

We can split the middle terms as follows:

$$x^2 - 9x + 18 = x^2 - 3x - 6x + 18$$

And now we factorize the terms in pairs:

$$x^2 - 3x - 6x + 18 = x(x - 3) - 6(x - 3) = (x - 3)(x - 6)$$

Note that we could've split the middle term as:

$$x^2 - 9x + 18 = x^2 - 6x - 3x + 18$$

This produces the same result:

$$x^2 - 6x - 3x + 18 = x(x - 6) - 3(x - 6) = (x - 6)(x - 3)$$

Exercise 2.2.5

Factorize the following expressions by splitting the middle term:

(a) $x^2 + 4x + 3$
(b) $x^2 + 7x + 10$
(c) $x^2 + 5x + 6$
(d) $x^2 - 8x + 12$
(e) $x^2 - 10x + 24$
(f) $x^2 - 9x + 20$
(g) $x^2 + 3x - 10$
(h) $x^2 - x - 20$
(i) $x^2 + 2x - 48$
(j) $x^2 + x - 42$
(k) $x^2 - 5x - 14$
(l) $x^2 + 3x - 28$
(m) $x^2 + (c + d)x + cd$
(n) $x^2 + (2e - 5f)x - 10ef$
(o) $x^2 + (t - 3s)x - 3st$

Both approaches can be used when the coefficient of x^2 is not 1 as in all previous examples. Consider:

$$(3x+2)(x+3) = 3x^2 + 9x + 2x + 6 = 3x^2 + 11x + 6$$

The constant term 6 comes from multiplying 2 and 3, the coefficient of x (11) comes from adding 3 times (coefficient of x in the first bracket) 3 and 2.

Worked example 2.2.6

Factorize $2x^2 + 5x + 3$:

We can look at numbers that multiplied together result in 3, but this time a sum one of the numbers and the twice other number should be 5:

factors	sum $(p+2q)$
1 and 3	7
3 and 1	5
-1 and -3	-7
-3 and -1	-5

Which gives:

$$2x^2 + 5x + 3 = (2x+1)(x+3)$$

Alternatively we can split the middle terms as follows:

$$2x^2 + 5x + 3 = 2x^2 + 2x + 3x + 3$$

And proceed as before:

$$2x^2 + 2x + 3x + 3 = 2x(x+1) + 3(x+1) = (x+1)(2x+3)$$

Exercise 2.2.6

Factorize the following expressions:

(a) $2x^2 - 5x + 2$ (b) $2x^2 + 3x - 5$ (c) $2x^2 + 11x + 14$

(d) $3x^2 - 8x + 4$ (e) $3x^2 - x - 4$ (f) $3x^2 + 16x + 5$

(g) $5x^2 - 13x + 6$ (h) $5x^2 - 14x - 3$ (i) $5x^2 + 12x + 4$

(j) $4x^2 - 4x - 3$ (k) $4x^2 + 5x - 6$ (l) $6x^2 + 11x - 10$

Consider the following cubic expression:

$$x^3 + 3x^2 - 4x - 12$$

We can factorize this expression by combining the methods we've used before - factoring terms in pairs and applying difference of squares:

$$x^3 + 3x^2 - 4x - 12 = x^2(x+3) - 4(x+3) = (x+3)(x^2-4) = (x+3)(x-2)(x+2)$$

Worked example 2.2.7a

Factorize $2x^3 + x^2 - 18x - 9$:

We can factor out x^2 from the first pair and -9 from the second pair:

$$2x^3 + x^2 - 18x - 9 = x^2(2x+1) - 9(2x+1)$$

Now we factor out $2x+1$ and apply difference of squares to $x^2 - 9$ to get:

$$(2x+1)(x^2-9) = (2x+1)(x-3)(x+3)$$

Some examples of quartic expressions can be factorized by applying the methods used for quadratics and difference of squares:

$$x^4 - 3x^2 + 2 = (x^2-1)(x^2-2) = (x-1)(x+1)(x-\sqrt{2})(x-\sqrt{2})$$

Worked example 2.2.7b

Factorize $x^4 - x^2 - 12$:

We first look for numbers that multiplied together give -12 and added together give -1 and we find that:

$$x^4 - x^2 - 12 = (x^2-4)(x^2+3)$$

The first bracket can be further factorized using difference of squares:

$$(x^2-4)(x^2+3) = (x-2)(x+2)(x^2+3)$$

Note that x^2+3 cannot be factorized any further, if we allow real coefficients only.

Exercise 2.2.7
Factorize the following expressions:

(a) $x^3 + 2x^2 - 9x - 18$

(b) $x^3 - 3x^2 - 5x + 15$

(c) $x^3 + x^2 - x - 1$

(d) $2x^3 + 3x^2 - 6x - 9$

(e) $3x^3 - x^2 - 6x + 2$

(f) $3x^3 + 2x^2 - 12x - 8$

(g) $x^3 - 5x^2 + x - 5$

(h) $4x^3 + 8x^2 - x - 2$

(i) $2x^3 - 6x^2 + x - 3$

(j) $x^3 + 7x^2 + 10x$

(k) $2x^3 - 11x^2 + 12x$

(l) $3x^3 + 14x^2 - 5x$

(m) $x^4 + 2x^3 - 4x^2 - 8x$

(n) $x^4 - 3x^3 - x^2 + 3x$

(o) $2x^4 + 3x^3 - 18x^2 - 27x$

(p) $x^4 - 5x^2 + 4$

(q) $x^4 + 2x^2 - 24$

(r) $x^4 - 12x^2 + 27$

(s) $4x^4 - 5x^2 + 1$

(t) $9x^4 - 10x^2 + 1$

(u) $2x^4 + 5x^2 - 3$

(v) $x^5 - 6x^3 + 8x$

(w) $x^5 + x^3 - 20x$

(x) $x^5 - 19x^3 + 90x$

ALGEBRAIC FRACTIONS

In order to add to fractions we need to make a common denominator:

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

Similarly if the fraction involves algebraic expressions:

$$\frac{1}{x}+\frac{1}{3}=\frac{3}{3x}+\frac{x}{3x}=\frac{3+x}{3x}$$

Note that because x appears in the denominator of one of the fractions, we must add the assumption that $x \neq 0$.

Worked example 2.2.8

Combine the following fractions to form a single fraction. State necessary assumptions.

$$\frac{2}{x}+\frac{1}{x+1}-\frac{1}{2}$$

The common denominator of all three fractions will be the product of their denominators, which is $2x(x+1)$:

$$\frac{4(x+1)}{2x(x+1)}+\frac{2x}{2x(x+1)}-\frac{x(x+1)}{2x(x+1)}$$

Now we can add and subtract the fractions to get:

$$\frac{4(x+1)+2x-x(x+1)}{2x(x+1)}=\frac{-x^2+5x+4}{2x(x+1)}$$

We need to make sure that 0 does not appear in the denominator, so $x \neq 0$ and $x \neq -1$.

Exercise 2.2.8

Combine the following fractions into a single fraction. State all necessary assumptions.

(a) $\frac{a}{3}+\frac{2}{a}$

(b) $\frac{1}{q+1}-\frac{2}{3}$

(c) $\frac{2}{t-1}+\frac{3}{t+1}$

(d) $\frac{1}{x}-\frac{1}{x-1}$

(e) $\frac{2}{y+1}+\frac{1}{y+2}$

(f) $\frac{1}{p}-\frac{1}{q}$

(g) $\frac{2}{p}+\frac{3}{q}-\frac{1}{pq}$

(h) $\frac{3}{x-1}+\frac{2}{x+1}-\frac{1}{3}$

(i) $2-\frac{3}{x+1}+\frac{1}{x}$

(j) $\frac{4}{x^2}-\frac{1}{x}+2$

(k) $\frac{2}{q}+\frac{q}{3}-q$

(l) $\frac{1}{x^2-4}+\frac{2}{x-2}-\frac{1}{x+2}$

An algebraic fraction can be simplified, if the numerator and denominator contain a common factor.

Worked example 2.2.9

Simplify the following fraction. State necessary assumptions.

$$\frac{2x^2 - x + 3}{x^2 - 4}$$

We can factorize both the denominator and the numerator:

$$\frac{2x^2 - x + 3}{x^2 - 4} = \frac{(2x+3)(x-2)}{(x-2)(x+2)}$$

Now we must assume that $x \neq 2$ and $x \neq -2$ in order to make sure that the denominator is not equal to 0. We can then simplify the fraction by dividing the denominator and numerator by $x - 2$, to get:

$$\frac{(2x+3)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)} = \frac{2x+3}{x+2}$$

Exercise 2.2.9

Simplify the following algebraic fractions. State all necessary assumptions.

(a) $\dfrac{x}{x^2 - 2x}$

(b) $\dfrac{x^2 - 9}{x^2 + 3x}$

(c) $\dfrac{x^2 + 5x}{x^2 - 25}$

(d) $\dfrac{x^2 - 16}{x^2 - 5x + 4}$

(e) $\dfrac{x^2 + x - 12}{x^2 - 9}$

(f) $\dfrac{2x^2 + x}{2x^2 - 3x - 2}$

(g) $\dfrac{x^2 + 3x - 18}{x^2 - 5x + 6}$

(h) $\dfrac{x^2 + 5x - 14}{x^2 + 8x + 7}$

(i) $\dfrac{x^2 + x - 20}{x^2 + 8x + 15}$

(j) $\dfrac{2x^2 + 3x - 2}{2x^2 + 7x - 4}$

(k) $\dfrac{3x^2 + x - 2}{3x^2 - x - 4}$

(l) $\dfrac{2x^2 - 9x + 10}{3x^2 - 11x + 10}$

(m) $\dfrac{x^2 - 2x}{x^3 + x^2 - 4x - 4}$

(n) $\dfrac{x^3 + 5x^2 - x - 5}{x^2 + 4x - 5}$

(o) $\dfrac{x^3 + x}{x^4 - 3x^2 - 4}$

(p) $\dfrac{x^2 - 4x + 4}{x^3 - 4x}$

(q) $\dfrac{x^3 + 3x^2 - x - 3}{x^3 + x^2 - 9x - 9}$

(r) $\dfrac{x^4 - 13x^2 + 36}{x^2 - x - 6}$

Sum and difference of cubes

1. Expand the following:

$$(a-b)(a^2+ab+b^2) \qquad \text{and} \qquad (a+b)(a^2-ab+b^2)$$

You should get the formulae for difference and sum of cubes:

$$a^3-b^3=(a-b)(a^2+ab+b^2) \qquad \text{and} \qquad a^3+b^3=(a+b)(a^2-ab+b^2)$$

2. Using the above formulae factorize:

$$x^3-1 \qquad \text{and} \qquad x^3+1$$

3. Factorize the following expressions

$8x^3-27$ $\qquad$ $64p^3-1$ $\qquad$ $4q^3-125$

a^3+8 $\qquad$ $27y^3+125$ $\qquad$ $64b^3+2a^3$

4. State the values of a and b for which the following is true:

$$(\sqrt[3]{2}-1)(a^2+ab+b^2)=(\sqrt[3]{2})^3-1^3=2-1=1$$

5. Rationalize the denominator of the expression $\dfrac{1}{\sqrt[3]{2}-1}$

6. Rationalize the denominator of the expression $\dfrac{1}{\sqrt[3]{2}+1}$

7. Rationalize the denominator of the following expressions:

$\dfrac{6}{\sqrt[3]{3}-1}$ $\qquad$ $\dfrac{8}{2-\sqrt[3]{2}}$ $\qquad$ $\dfrac{\sqrt{2}}{3-\sqrt{3}[2]}$

$\dfrac{5}{\sqrt[3]{3}+1}$ $\qquad$ $\dfrac{\sqrt{5}}{\sqrt[3]{5}+2}$ $\qquad$ $\dfrac{\sqrt[3]{4}}{\sqrt[3]{2}+\sqrt[3]{3}}$

SHORT TEST

1. [4 *points*]

Factorize the following expressions:

(a) $4x^2 - 25$

(b) $x^2 + 6x - 7$

(c) $2x^2 + 5x - 12$

(d) $x^3 + 5x^2 - 4x - 20$

2. [3 *points*]

Write the following expressions as a single fraction (simplify your answer if possible):

(a) $\frac{p}{2} - \frac{3}{p}$

(b) $q + \frac{2}{q} - \frac{3}{q+1}$

3. [6 *points*]

Simplify the following algebraic fractions, state all necessary assumptions:

(a) $\frac{x^2 - 9}{x^2 - 3x}$

(b) $\frac{x^2 - 3x - 10}{x^2 + 3x + 2}$

SHORT TEST

SOLUTIONS

1. [4 *points*]

Factorize the following expressions:

(a) $4x^2 - 25$

$= (2x-5)(2x+5)$

(b) $x^2 + 6x - 7$

$= (x+7)(x-1)$

(c) $2x^2 + 5x - 12$

$= (2x-3)(x+4)$

(d) $x^3 + 5x^2 - 4x - 20$

$= (x+5)(x-2)(x+2)$

2. [3 *points*]

Write the following expressions as a single fraction (simplify your answer if possible):

(a) $\frac{p}{2} - \frac{3}{p} = \frac{p^2-6}{2p}$

(b) $q + \frac{2}{q} - \frac{3}{q+1} = \frac{q^3+q^2-q+2}{q(q+1)}$

3. [6 *points*]

Simplify the following algebraic fractions, state all necessary assumptions:

(a) $\frac{x^2-9}{x^2-3x} = \frac{(x-3)(x+3)}{x(x-3)} = \frac{x+3}{x}$

Assumptions $x \neq 0$ and $x \neq 3$.

(b) $\frac{x^2-3x-10}{x^2+3x+2} = \frac{(x-5)(x+2)}{(x+1)(x+2)} = \frac{x-5}{x+1}$

Assumptions $x \neq -1$ and $x \neq -2$.

1.3 Simultaneous linear equations

LINEAR EQUATIONS

A linear equation is an equation where the highest power of the variable (unknown) is 1. The following are linear equations:

$$3x + 2 = 5 \qquad 2x - 4y + 7 = 0 \qquad \frac{1}{2}x - \sqrt{3}y + \pi z = 1$$

Note that the first equation is a linear equation in one variable, the second in two variables and the third in three variables. A solution to a linear equation are the values that when substituted for the unkowns make the equality true.

Worked example 2.3.1

Decide if the following values are solutions to the linear equation:

$$4x - 7 = 5$$

(a) 1 (b) 2 (c) 3 (d) 4

(a) 1 is not a solution since:

$$4 \times 1 - 7 \neq 5$$

(b) 2 is not a solution since:

$$4 \times 2 - 7 \neq 5$$

(c) 3 is a solution since:

$$4 \times 3 - 7 = 5$$

(d) 4 is not a solution since:

$$4 \times 4 - 7 \neq 5$$

Solving a linear equation in one variable requires performing operations (adding, subtracting, multiplying and dividing) on both sides of the equation to reach the point where the unknown is left by itself on one of the sides of the equation.

Worked example 2.3.2a

Solve the equation:

$$4x - 7 = 5$$

We add 7 to both sides of the equation to get:

$$4x = 12$$

Now we divide both sides by 4 to get the solution:

$$x = 3$$

Worked example 2.3.2b

Solve the equation:

$$2x - 1 = x\sqrt{3} + 2$$

We add 1 and subtract $x\sqrt{3}$ to get:

$$2x - x\sqrt{3} = 3$$

Now we can factor out x on the left hand side to get:

$$x(2 - \sqrt{3}) = 3$$

We divide by $(2 - \sqrt{3})$:

$$x = \frac{3}{2 - \sqrt{3}}$$

Finally (this step may not be required) we rationalize the denominator:

$$x = \frac{3}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$

which gives:

$$x = 6 + 3\sqrt{3}$$

Exercise 2.3.2

Solve the following equations:

(a) $5x - 7 = 2x + 3$

(b) $3y - 4 = y + 11$

(c) $4z + 7 = 2z - 6$

(d) $6a - 9 = 3a + 2$

(e) $8b + 7 = 4b + 12$

(f) $2c - 3 = 5c + 5$

(g) $d\sqrt{3} + 5 = 3d - 1$

(h) $7e + 9 = e\sqrt{2} - 14$

(i) $5f - 6 = f\sqrt{7} + 3$

(j) $g\sqrt{5} + 2 = 4g - 7$

(k) $3h - 8 = h\sqrt{11} + 1$

(l) $11x + 4 = x\sqrt{2} - 9$

(m) $y\sqrt{5} - 3 = 6y + 5$

(n) $2x - 11 = x\sqrt{3} + 3$

(o) $4q\sqrt{2} + 7 = 31 - \sqrt{2}$

(p) $6n - \sqrt{3} = n\sqrt{3} + 9$

(q) $9p + 3 = p\sqrt{2} + \sqrt{3}$

(r) $2q - \sqrt{3} = 2q\sqrt{5} + 12$

(s) $(x + 1)^2 + (x - 3)(x + 3) = 2(x - 2)^2$

(t) $(x - 2)^2 + (x - 3)^2 = 2(x - 1)(x + 1)$

(u) $2(x - 1)^2 + (4 - x)(4 + x) = (x + 5)^2$

(v) $2(x + 2)^2 - (x - 5)(x + 5) = (x - \sqrt{3})^2$

SIMULTANEOUS LINEAR EQUATIONS

Consider the following pair of linear equations:

$$\begin{cases} 3x + y = 1 \\ 2x - 3y = 19 \end{cases}$$

The pair $x = 0$ and $y = 1$ satisfies the first equation, but it does not satisfy the second equation, so it is not a solution to this system of equation. The pair $x = 2$ and $y = -5$ satisfies both equations and therefore it is the solution to the system.

Worked example 2.3.3

Solve the system of equations:

$$\begin{cases} 3x + y = 1 \\ 2x - 3y = 19 \end{cases}$$

Method 1 - substitution

We will use one of the equations to express one of the variables in terms of the other. In this example we will use the first equation to express y in terms of x:

$$y = 1 - 3x$$

We will substitute this expression for y in the other equation:

$$2x - 3(1 - 3x) = 19$$

Now we have a linear equation with one variable. We solve it to get $x = 2$. We still need to find y:

$$y = 1 - 3 \times 2 = -5$$

So the solution to the system is the pair $x = 2$ and $y = -5$.

Method 2 - elimination

We can try to eliminate one of the variables by combining the two equations. In our example we will first multiply the first equation by 3 to get:

$$\begin{cases} 9x + 3y = 3 \\ 2x - 3y = 19 \end{cases}$$

Now we can add the two equations together. Note that $3y$ will cancel with $-3y$ and we will be left with a single variable:

$$11x = 22$$

Which gives $x = 2$. We still need to find y. Going back to one of the original equations:

$$3 \times 2 + y = 1$$

So $y = -5$ and we found that the pair $x = 2$ and $y = -5$ is the solution to the system.

Exercise 2.3.3a

Solve the following systems of equations:

(a) $\begin{cases} 3x + 4y = 12 \\ 2x - y = 5 \end{cases}$

(b) $\begin{cases} x - 2y = 1 \\ 3x + y = 7 \end{cases}$

(c) $\begin{cases} 5x + 2y = 10 \\ x + 3y = 6 \end{cases}$

(d) $\begin{cases} 2x + 3y = 9 \\ 4x - y = 8 \end{cases}$

(e) $\begin{cases} x + 5y = 11 \\ 2x + 4y = 10 \end{cases}$

(f) $\begin{cases} 3x - y = 2 \\ 5x + 2y = 17 \end{cases}$

(g) $\begin{cases} x + y = 5 \\ 2x - 3y = 4 \end{cases}$

(h) $\begin{cases} 4x + 6y = 24 \\ x - y = 2 \end{cases}$

(i) $\begin{cases} 2x + y = 7 \\ 3x + 4y = 18 \end{cases}$

(j) $\begin{cases} 6x - 2y = 10 \\ 3x + y = 8 \end{cases}$

(k) $\begin{cases} 5x + 3y = 20 \\ 2x - y = 3 \end{cases}$

(l) $\begin{cases} x - 3y = -4 \\ 4x + 2y = 14 \end{cases}$

(m) $\begin{cases} \frac{1}{2}x + \frac{3}{4}y = 5 \\ \frac{3}{4}x - y = \frac{1}{2} \end{cases}$

(n) $\begin{cases} x - \frac{2}{3}y = \frac{3}{5} \\ \frac{1}{4}x + y = 2 \end{cases}$

(o) $\begin{cases} \frac{2}{5}x + y = \frac{7}{3} \\ x + \frac{4}{3}y = 3 \end{cases}$

(p) $\begin{cases} \frac{3}{7}x + y = 4 \\ 3x - \frac{1}{2}y = 5 \end{cases}$

(q) $\begin{cases} x + \frac{1}{3}y = \frac{5}{2} \\ \frac{1}{3}x - y = 2 \end{cases}$

(r) $\begin{cases} \frac{5}{6}x + \frac{2}{3}y = 1 \\ \frac{2}{3}x + y = 1 \end{cases}$

(s) $\begin{cases} \sqrt{2}x - y = 3 \\ x + \sqrt{3}y = 2 \end{cases}$

(t) $\begin{cases} \sqrt{5}x + y = 4 \\ 3x - \sqrt{2}y = 1 \end{cases}$

(u) $\begin{cases} x + \sqrt{7}y = 5 \\ \sqrt{3}x - y = 6 \end{cases}$

(v) $\begin{cases} 2x - \sqrt{5}y = 4 \\ \sqrt{2}x + y = \sqrt{3} \end{cases}$

(w) $\begin{cases} \sqrt{3}x + y = 6 \\ x - \sqrt{6}y = 2 \end{cases}$

(x) $\begin{cases} \sqrt{7}x + \sqrt{2}y = 3 \\ 2x - y = \sqrt{5} \end{cases}$

(y) $\begin{cases} \sqrt{11}x + 2y = 5 \\ 3x - \sqrt{3}y = 7 \end{cases}$

(z) $\begin{cases} 2\sqrt{2}x + y = 2 \\ x + \sqrt{6}y = 4 \end{cases}$

(zz) $\begin{cases} 4x - \sqrt{2}y = 9 \\ \sqrt{5}x + y = 3 \end{cases}$

Exercise 2.3.3b
Solve the following systems of equations, write your answer in terms of parameter k:

(a) $\begin{cases} 2x + 3y = 2k \\ x - y = 1 \end{cases}$

(b) $\begin{cases} 3x - y = k \\ 2x + 5y = k + 1 \end{cases}$

(c) $\begin{cases} x + 5y = 1 - k \\ 3x - y = 2k \end{cases}$

(d) $\begin{cases} 2x - 3y = k + 1 \\ 4x + y = k - 1 \end{cases}$

(e) $\begin{cases} 3x + 5y = 1 - k \\ x - 2y = 2k \end{cases}$

(f) $\begin{cases} 3x - 4y = 1 - 3k \\ 2x + 3y = k \end{cases}$

Exercise 2.3.3c
Solve the following systems of equations:

(a) $\begin{cases} \dfrac{x+1}{3} - \dfrac{y-2}{4} = 2 \\ \dfrac{2-x}{3} + \dfrac{3-y}{5} = 1 \end{cases}$

(b) $\begin{cases} \dfrac{2x-1}{5} - \dfrac{y-1}{2} = 1 \\ \dfrac{4-x}{2} = \dfrac{y}{5} - 3 \end{cases}$

(c) $\begin{cases} \dfrac{x}{3} - \dfrac{y}{2} = \dfrac{x-2}{4} \\ \dfrac{y-4}{3} + 2x = -1 \end{cases}$

(d) $\begin{cases} x + 2 - \dfrac{y-1}{3} = \dfrac{2-x}{4} \\ \dfrac{y+4}{2} - \dfrac{x}{2} = \dfrac{x-5y}{4} \end{cases}$

(e) $\begin{cases} y - \dfrac{2y-1}{5} = \dfrac{x+1}{4} \\ \dfrac{x+y}{5} + y = \dfrac{3y+x}{4} + 1 \end{cases}$

(f) $\begin{cases} \dfrac{x}{3} + \dfrac{2-y}{2} = \dfrac{3x+1}{4} \\ y + \dfrac{x}{3} = \dfrac{x+2y}{5} - \dfrac{2x+y}{2} + 1 \end{cases}$

Exercise 2.3.3d
Solve the following systems of equations:

(a) $\begin{cases}(x-1)^2+2y=(x-3)(x+3)\\ y^2+2x-1=(y-1)^2\end{cases}$

(b) $\begin{cases}(x-2)^2-y=(x-1)^2\\ y^2+3x=(y-3)^2\end{cases}$

(c) $\begin{cases}(y-1)^2-y=x+y^2\\ (x+2)^2=(x-1)(x+4)+y\end{cases}$

(d) $\begin{cases}(y+2)^2-(y-1)^2=3x+3\\ (x+1)^2=(x-2)(x+2)+2y\end{cases}$

(e) $\begin{cases}(y+3)^2-(y-2)^2=x\\ (2x+1)^2+11=(2x-3)(2x+3)-y\end{cases}$

(f) $\begin{cases}(x-2)(x+1)-y=x^2\\ (y-3)(y+3)=(1+y)^2-x\end{cases}$

(g) $\begin{cases}(x-3)(x+2)-y=(x+1)^2\\ y(y-3)=(1-y)^2+x\end{cases}$

(h) $\begin{cases}(2x-3)^2-y=4(x+1)^2\\ y(1-y)-16=2x-(1+y)^2\end{cases}$

(i) $\begin{cases}(x+1)(y-3)=(x+2)(y-2)\\ (x-3)(y-2)=(x+1)(y-4)\end{cases}$

(j) $\begin{cases}(x+2)(y-2)=(x+3)(y-1)\\ (x-2)(y-1)=(x+2)(y-3)\end{cases}$

(k) $\begin{cases}(x-2)^2-y=x(x+3)\\ (x+2)(y+3)=(x-1)(y-3)\end{cases}$

(l) $\begin{cases}(y-3)^2-2x=(y-1)(y+1)\\ (x+1)(y+2)=(x-2)(y-1)\end{cases}$

Consider the following system:

$$\begin{cases} 2x + 3y = 3 \\ 4x + 6y = 5 \end{cases}$$

Solving this system will lead to a contradiction $0 = 1$. This means that this system is **inconsistent**, it does not have any solutions. On the other hand the system:

$$\begin{cases} 2x + 3y = 3 \\ 4x + 6y = 6 \end{cases}$$

is **consistent**, because it has a solution. But this system has in fact infinitely many solutions. Solving the system leads to $0 = 0$. Note that the fact that a system has infinitely many solutions does not mean that every pair of numbers is a solution. For the above system we can write the solutions in the form $x = \lambda$ and $y = 1 - \frac{2}{3}\lambda$ where λ can be any real number. So for example $x = 3$ and $y = -1$ is a solution, but also $x = 4$ and $y = -\frac{5}{3}$ etc.

Worked example 2.3.4

For the following system of equations:

$$\begin{cases} 2x + 5y = 12 \\ x + ay = b \end{cases}$$

Find the values of a and b for which this system is:

(a) consistent with 1 solution,

(b) consistent with infinitely many solutions,

(c) inconsistent.

We start solving the system. We will use elimination methods, so we will multiply the second equation by 2:

$$\begin{cases} 2x + 5y = 12 \\ 2x + 2ay = 2b \end{cases}$$

Now we subtract the first equation from the second one in order to eliminate x:

$$2ay - 5y = 2b - 12$$

We factor out y:

$$(2a - 5)y = 2b - 12$$

Now note that if $2a - 5 \neq 0$, we can divide both sides by $2a - 5$ and find that $y = \dfrac{2b - 12}{2a - 5}$. If we now substitute this into the second equation and solve for x we get that the system has a unique solution $x = \dfrac{12a - 5b}{2a - 5}$ and $y = \dfrac{2b - 12}{2a - 5}$.

Now if $a = 2.5$, the left hand side of the equation is 0. Then if $2b - 12 = 0$ (so if $b = 6$), the right hand side is also 0 and we get $0 = 0$, a consistent system with infinitely many solutions. However if $a = 2.5$, but $b \neq 6$, then we get a contradiction when we solve the system, meaning that it is inconsistent.

(a) $a \neq 2.5$.

(b) $a = 2.5$ and $b = 6$.

(c) $a = 2.5$ and $b \neq 6$.

Exercise 2.3.4
For the following systems of equations find the values of a and b for which this system is:

(i) consistent with 1 solution,

(ii) consistent with infinitely many solutions,

(iii) inconsistent.

(a) $\begin{cases} x+3y=b \\ ax-6y=5 \end{cases}$ (b) $\begin{cases} 2x-3y=6 \\ 6x+ay=b \end{cases}$ (c) $\begin{cases} 4x+y=3 \\ x+ay=b \end{cases}$

(d) $\begin{cases} 3x+2y=b \\ 2ax-y=8+b \end{cases}$ (e) $\begin{cases} 2x-y=b+1 \\ 2x+(a+3)y=7-3b \end{cases}$ (f) $\begin{cases} 4x-5y=2b \\ 5x+(a-1)y=1+b \end{cases}$

When solving linear system with 3 equations and 3 unknowns both elimination and substitution methods can be applied.

Worked example 2.3.5

Solve the system of equations:

$$\begin{cases} 2x+y+z=2 \\ 2x-y-2z=15 \\ 3x-3y+2z=-4 \end{cases}$$

Method 1 - substitution

We can use the first equation to express z in terms of x and y:

$$z=2-2x-y$$

We will substitute this expression for z into the other two equations:

$$\begin{cases} 2x-y-2(2-2x-y)=15 \\ 3x-3y+2(2-2x-y)=-4 \end{cases}$$

Which simplifies to:

$$\begin{cases} 6x+y=19 \\ -x-5y=-8 \end{cases}$$

Now we have a system of two equations and two unknowns. We solve this to get $x=3$ and $y=1$, which gives:

$$z=2-2\times 3-1=-5$$

So the solution to the system is a triple $x=3$, $y=1$ and $z=-5$.

$$\begin{cases} 2x + y + z = 2 & \text{①} \\ 2x - y - 2z = 15 & \text{②} \\ 3x - 3y + 2z = -4 & \text{③} \end{cases}$$

Method 2 - elimination

We can use one of the equations to eliminate one of the variables from the other two equations. In this example we can add the second equation to the third one and then the first equation to the second equation twice. This gives:

$$\begin{cases} 6x + y = 19 & \text{②} + 2 \times \text{①} \\ 5x - 4y = 11 & \text{③} + \text{②} \end{cases}$$

Solving the above gives $x = 3$ and $y = 1$. Now we come back to one of the original equations:

$$2 \times 3 + 1 + z = 2$$

which gives $z = -5$. So we get $x = 3$, $y = 1$ and $z = -5$.

Exercise 2.3.5
Solve the following systems of equations:

(a) $\begin{cases} x + 2y + 3z = 4 \\ 4x - y + z = 14 \\ 3x + 5y - 2z = -1 \end{cases}$

(b) $\begin{cases} 2x - y + 4z = 19 \\ 3x + 2y + z = 4 \\ 5x - 3y + 2z = 3 \end{cases}$

(c) $\begin{cases} x + y + z = -3 \\ 2x - y + 3z = 8 \\ 4x + 5y - z = 12 \end{cases}$

(d) $\begin{cases} 3x + y - 4z = -14 \\ 2x + 3y + 5z = 22 \\ x - y + z = -7 \end{cases}$

(e) $\begin{cases} x - 2y + z = -6 \\ 4x + y - 3z = 15 \\ 2x + 3y + z = 10 \end{cases}$

(f) $\begin{cases} 5x + 2y - 3z = 7 \\ 2x + 4y + z = 20 \\ 3x - y + z = 2 \end{cases}$

(g) $\begin{cases} x + 4y + z = 3 \\ 3x - 2y + 5z = 25 \\ 2x + y - 3z = -3 \end{cases}$

(h) $\begin{cases} 2x + y + 3z = 18 \\ 4x - y + z = 13 \\ 5x + 2y - 4z = 7 \end{cases}$

(i) $\begin{cases} 3x + 2y + z = -6 \\ x - 2y + 5z = 11 \\ 4x + y = 9 \end{cases}$

(j) $\begin{cases} x + y + 2z = 12 \\ 2x - y - z = -11 \\ 3x + 4y + z = 5 \end{cases}$

(k) $\begin{cases} x - y + 3z = -2 \\ 4x + 2y - z = 15 \\ 2x + y + 5z = 7 \end{cases}$

(l) $\begin{cases} 2x + 3y - z = 12 \\ 3x - 4y + z = -10 \\ x + 2y + 3z = 8 \end{cases}$

(m) $\begin{cases} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = \frac{1}{5} \\ \frac{2}{3}x - \frac{1}{2}y + z = \frac{3}{4} \\ x + \frac{3}{4}y - \frac{1}{5}z = \frac{2}{3} \end{cases}$

(n) $\begin{cases} \frac{3}{4}x - \frac{1}{5}y + z = 1 \\ \frac{1}{3}x + \frac{2}{5}y - \frac{1}{2}z = \frac{1}{4} \\ x - \frac{3}{5}y + \frac{2}{3}z = \frac{5}{6} \end{cases}$

(o) $\begin{cases} \frac{1}{6}x + y - \frac{1}{3}z = \frac{1}{2} \\ \frac{5}{8}x + \frac{1}{4}y + \frac{2}{3}z = \frac{3}{5} \\ x + \frac{1}{3}y - \frac{1}{2}z = \frac{2}{7} \end{cases}$

Crammer's rule

A matrix is a rectangular array of numbers. The following are examples of matrices:

$$\begin{pmatrix} 5 & -2 & 1 \\ 3 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 2 & -5 \\ 3 & 0 \\ -7 & 6 \end{pmatrix} \qquad \begin{pmatrix} 3 & 4 \\ 5 & -7 \end{pmatrix}$$

An n by m matrix (or $n \times m$ matrix) is a matrix with n rows and m columns. The list above consists of a 2 by 3, 4 by 2 and 2 by 2 matrices.

A determinant of a 2 by 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

1. Calculate the determinant of the following matrices:

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix} \qquad \begin{pmatrix} 3 & 2 \\ 9 & 6 \end{pmatrix}$$

2. Show that a system:

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

has a unique solution $x = \dfrac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}, \quad y = \dfrac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$ provided $\begin{vmatrix} a & b \\ d & e \end{vmatrix} \neq 0$

3. Solve the following systems using the above method:

$$\begin{cases} 2x + 3y = 4 \\ 3x + 4y = 5 \end{cases} \qquad \begin{cases} 5x + 2y = 1 \\ 4x - 7y = 11 \end{cases} \qquad \begin{cases} 3x + 5y = -2 \\ 2x + 6y = -3 \end{cases}$$

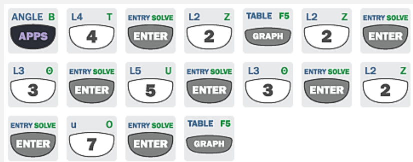
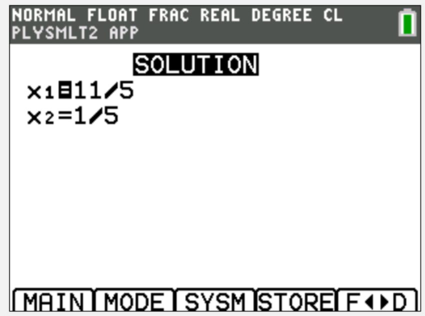
4. Explain what can we say about the system:

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

in the case $\begin{vmatrix} a & b \\ d & e \end{vmatrix} = 0$

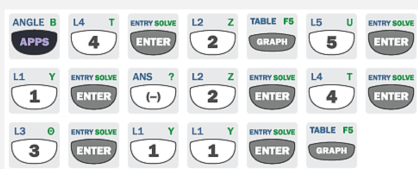
Simultaneous linear equations can be solved on GDC using the PlySmlt2 APP.

EXAMPLE 1

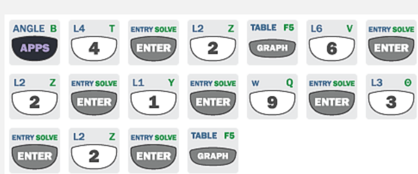
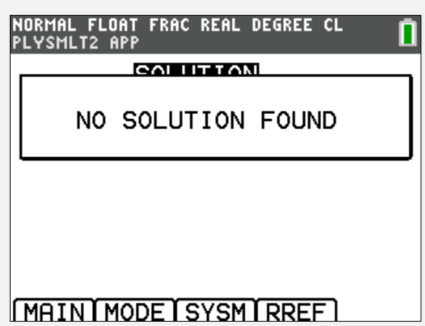
Solve:	**INPUT**	**OUTPUT**
$\begin{cases} 2x + 3y = 5 \\ 3x + 2y = 7 \end{cases}$		

Note that the interface of the APP is different (and more intuitive) in the newer versions of the GDC. You do not input the coefficients into a matrix, but into a system of equations.

EXAMPLE 2

Solve:	**INPUT**	**OUTPUT**
$\begin{cases} 5x + y = -2 \\ 4x + 3y = 11 \end{cases}$		

EXAMPLE 3

Solve:	**INPUT**	**OUTPUT**
$\begin{cases} 6x + 2y = 1 \\ 9x + 3y = 2 \end{cases}$		

Note that in the last example we have a system which is inconsistent, so the calculator won't find any solutions.

SHORT TEST

1. [3 *points*]

Solve the following equations:

(a) $7x + 13 = 4x - 5$

(b) $3x\sqrt{2} - 1 = 2x + \sqrt{2}$

2. [5 *points*]

Solve the following systems of equations:

(a) $\begin{cases} 3x + 2y = 7 \\ 2x + 5y = -10 \end{cases}$

(b) $\begin{cases} x + y + 2z = 9 \\ 2x + 3y - z = 0 \\ 3x - y + z = -1 \end{cases}$

3. [4 *points*]

For the following system of equations find the values of a and b for which this system is:

(i) consistent with 1 solution,

(ii) consistent with infinitely many solutions,

(iii) inconsistent.

$$\begin{cases} 2x - y = b \\ 6x + ay = 15 \end{cases}$$

SHORT TEST

SOLUTIONS

1. [3 *points*]

Solve the following equations:

(a) $7x + 13 = 4x - 5$

$3x = -18$

$x = -6$

(b) $3x\sqrt{2} - 1 = 2x + \sqrt{2}$

$3x\sqrt{2} - 2x = \sqrt{2} + 1$

$x(3\sqrt{2} - 2) = \sqrt{2} + 1$

$$x = \frac{\sqrt{2} + 1}{3\sqrt{2} - 2} = \frac{5\sqrt{2} + 8}{14}$$

2. [5 *points*]

Solve the following systems of equations:

(a) $\begin{cases} 3x + 2y = 7 \\ 2x + 5y = -10 \end{cases}$

$\begin{cases} 15x + 10y = 35 \\ 4x + 10y = -20 \end{cases}$

$11x = 55$

$x = 5,\ y = -4$

(b) $\begin{cases} x + y + 2z = 9 \\ 2x + 3y - z = 0 \\ 3x - y + z = -1 \end{cases}$

$\begin{cases} 5x + 7y = 9 & ① + 2 \times ② \\ 5x + 2y = -1 & ③ + ② \end{cases}$

$5y = 10$

$x = -1, y = 2, z = 4$

3. [4 *points*]

For the following system of equations find the values of a and b for which this system is:

(i) consistent with 1 solution,

(ii) consistent with infinitely many solutions,

(iii) inconsistent.

$$\begin{cases} 2x - y = b \\ 6x + ay = 15 \end{cases}$$

$$\begin{cases} 6x - 3y = 3b \\ 6x + ay = 15 \end{cases}$$

$$(a + 3)y = 15 - 3b$$

(i) $a \neq -3$,

(ii) $a = -3$ and $b = 5$,

(iii) $a = -3$ and $b \neq 5$.

1.4 Quadratic equations

A quadratic equation is an equation that can be arranged in the form:

$$ax^2 + bx + c = 0$$

where x is the unknown variable and a, b and c are known numbers (called coefficients) with $a \neq 0$. Note that when $a = 0$, then the term ax^2 disappears and we're left with a linear equation $bx + c = 0$. The following are examples of quadratic equations:

$$2x^2 + x - 3 = 0 \qquad -\frac{1}{2}x^2 - 1 = 0 \qquad 11x^2 + \sqrt{2}x = 0$$

Note that in the first example we have $a = 2, b = 1$ and $c = -3$. In the second example we have $a = -\frac{1}{2}, b = 0$ and $c = -1$ and finally in the third example we have $a = 11, b = \sqrt{2}$ and $c = 0$.

In this section we will learn three methods for solving quadratic equation.

FACTORIZATION

The factorization method is based on a useful property that if $a \times b = 0$, then at least one of a and b must be 0. This property can be applied to an equation in the following (factorized) form:

$$(x - 7)(x + 2) = 0$$

We have a product of $(x - 7)$ and $(x + 2)$ and this product is equal to 0, so we must have $x - 7 = 0$ or $x + 2 = 0$. This means that $x = 7$ or $x = -2$ and these are our solutions to the above equation.

Some special cases of quadratic equations are especially easy to solve. Let's first consider the case where $c = 0$.

Worked example 2.4.1

Solve:

$$3x^2 - 8x = 0$$

We can factor out x and get:

$$x(3x - 8) = 0$$

Now a product of two terms is 0, so at least one of these must be 0. We must have $x = 0$ or $3x - 8 = 0$. So the solutions are $x = 0$ or $x = \frac{8}{3}$.

Exercise 2.4.1
Solve the following equations by factoring x out:

(a) $2x^2 - 5x = 0$
(b) $4x^2 - x = 0$
(c) $6x^2 + 5x = 0$
(d) $3x^2 = 4x$
(e) $x^2 = -7x$
(f) $\frac{1}{2}x^2 + \frac{1}{3}x = 0$
(g) $\frac{2}{3}x^2 - \frac{3}{4}x = 0$
(h) $2x^2 - \sqrt{3}x = 0$
(i) $\sqrt{2}x^2 + 2x = 0$
(j) $\frac{1}{3}x^2 = 2x$
(k) $\frac{3}{4}x^2 = \sqrt{2}x$
(l) $\sqrt{5}x^2 = \sqrt{3}x$

Another easy case occurs when $b = 0$. Our expression can then be factorized using difference of squares formula. Recall that we have $a^2 - b^2 = (a - b)(a + b)$.

Worked example 2.4.2

Solve:

$$4x^2 - 9 = 0$$

We rewrite the left hand side as a difference of squares:

$$(2x)^2 - 3^2 = 0$$

and apply the difference of squares formula:

$$(2x - 3)(2x + 3) = 0$$

Now $2x - 3 = 0$ or $2x + 3 = 0$, so $x = \frac{3}{2}$ or $x = -\frac{3}{2}$.

Exercise 2.4.2
Solve the following equations using difference of squares:

(a) $x^2 - 16 = 0$
(b) $9x^2 - 25 = 0$
(c) $4x^2 - 49 = 0$
(d) $x^2 - 5 = 0$
(e) $3x^2 - 1 = 0$
(f) $36x^2 - 7 = 0$
(g) $49x^2 = 100$
(h) $2x^2 = \frac{1}{4}$
(i) $\frac{1}{2}x^2 = \frac{2}{3}$
(j) $4x^2 = \frac{1}{36}$
(k) $9x^2 = 2$
(l) $2x^2 = 9$

When neither b nor c is 0, we still want to factorize the quadratic. This can be done using the methods from section 2.

Worked example 2.4.3

Solve:

$$x^2 - 3x - 18 = 0$$

We factorize the expression $x^2 - 3x - 18$ into $(x-6)(x+3)$, which gives:

$$(x-6)(x+3) = 0$$

So either $x - 6 = 0$, which gives $x = 6$, or $x + 3 = 0$, which gives $x = -3$. The solutions are $x = 6$ or $x = -3$.

Exercise 2.4.3

Solve the following equations using factorization:

(a) $x^2 - 5x + 6 = 0$

(b) $x^2 + 4x + 3 = 0$

(c) $x^2 - 7x + 10 = 0$

(d) $x^2 + 6x + 9 = 0$

(e) $x^2 - 3x - 10 = 0$

(f) $x^2 + 2x - 15 = 0$

(g) $3x^2 + x - 2 = 0$

(h) $2x^2 - 5x + 3 = 0$

(i) $4x^2 - 4x - 3 = 0$

(j) $2x^2 + 3x - 2 = 0$

(k) $5x^2 - 11x + 6 = 0$

(l) $6x^2 - 5x - 4 = 0$

(m) $7x^2 + 10x + 3 = 0$

(n) $3x^2 + 7x + 2 = 0$

(o) $8x^2 - 14x + 3 = 0$

(p) $4x^2 + 6 = 10x$

(q) $9x^2 + 6 = 15x$

(r) $5x^2 + 9x + 5 = 1$

(s) $2x^2 + 6x = x + 3$

(t) $3x^2 + 1 = 8x - 3$

(u) $2x^2 + 2x = x + 1$

(v) $x^2 + 5x = \frac{1}{2}x + \frac{5}{2}$

(w) $x^2 - 8x + 9 = \frac{1}{3}x - \frac{1}{3}$

(x) $x^2 + 2.6x = 1.2$

COMPLETING THE SQUARE

Another approach is to **complete the square** by using the formulae for square of a sum or difference. Consider the equation:

$$x^2 - 4x + 3 = 0$$

By adding 1 to both sides we get:

$$x^2 - 4x + 4 = 1$$

which gives:

$$(x-2)^2 = 1$$

Now $(x-2)$ squared results in 1, so $x-2$ must be either 1 or -1. If $x-2=1$, then $x=3$. If $x-2=-1$, then $x=1$. The solutions to the original equation are $x=3$ or $x=1$.

There are 2 natural questions one can ask at this point: (1) how do we know what to add to complete the square and (2) what if the coefficient of x^2 is not 1? Let's start with the second question - as we are dealing with an equation it is always possible to divide both sides by the coefficient of x^2 reducing it to 1. As for the second question - the number the is added to/subtracted from in the bracket is always half of the coefficient of x (provided the coefficient of x^2 is 1) which means that we want the constant term to be half of the coefficient of x all squared.

Worked example 2.4.4a

Solve the following equation by completing the square:

$$2x^2 + 6x + 1 = 0$$

We start by dividing both sides of the equation by 2 in order to make the coefficient of x^2 equal to 1:

$$x^2 + 3x + \frac{1}{2} = 0$$

Now we know that the bracket will be $(x + \frac{3}{2})^2$, which means that the constant term needs to be $(\frac{3}{2})^2 = \frac{9}{4}$. We have $\frac{1}{2} = \frac{2}{4}$, so we need to add $\frac{7}{4}$ to both sides:

$$x^2 + 3x + \frac{9}{4} = \frac{7}{4}$$

$$\left(x + \frac{3}{2}\right)^2 = \frac{7}{4}$$

This gives:

$$x + \frac{3}{2} = \pm\frac{\sqrt{7}}{2}$$

and finally we get:

$$x = \frac{-3 \pm \sqrt{7}}{2}$$

Worked example 2.4.4b

Solve the following equation by completing the square:

$$x^2 + 8x + 21 = 0$$

The coefficient of x^2 is already 1. The bracket will be $(x+4)^2$, so the constant term needs to be $4^2 = 16$, which means that we need to subtract 5 from both sides:

$$x^2 + 8x + 16 = -5$$

$$(x + 4)^2 = -5$$

We can stop at this point and state that there are **no real solutions** to the above equation as no real number squared produces a negative number.

Exercise 2.4.4
Solve the following equations by completing the square:

(a) $x^2 - 4x + 3 = 0$ (b) $x^2 + 6x + 8 = 0$ (c) $x^2 - 8x + 15 = 0$

(d) $x^2 + 5x - 6 = 0$ (e) $x^2 - 3x + 2 = 0$ (f) $x^2 + 7x + 10 = 0$

(g) $2x^2 + 4x - 3 = 0$ (h) $2x^2 + 8x + 6 = 0$ (i) $2x^2 + 12x + 16 = 0$

(j) $3x^2 - x - 2 = 0$ (k) $4x^2 + 2x - 7 = 0$ (l) $5x^2 - 9x + 8 = 0$

(m) $x^2 - 2x + 3 = 0$ (n) $6x^2 + x - 5 = 0$ (o) $x^2 + x + 1 = 0$

(p) $5x^2 + 4x - 2 = 0$ (q) $x^2 - 4x + 7 = 0$ (r) $2x^2 + x + \sqrt{3} = 0$

(s) $x^2 - 6x - 2 = 0$ (t) $3x^2 + 2x - 5 = 0$ (u) $x^2 + 5x - 1 = 0$

(v) $2x^2 - x + 7 = 0$ (w) $2x^2 + 2x - 9 = 0$ (x) $4x^2 - 7x + 3 = 0$

(y) $x^2 + 3x + \frac{1}{2} = 0$ (z) $5x^2 - \frac{1}{2}x + 1 = 0$ (zz) $2x^2 + \frac{1}{3}x - 4 = 0$

QUADRATIC FORMULA

If we try to solve a general quadratic equation:

$$ax^2 + bx + c = 0$$

by completing the square, we will start by dividing both sides of the equation by the coefficient of x^2 to get:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Now we know that the bracket will be $(x + \frac{b}{2a})^2$, which means that the constant term should be $\frac{b^2}{4a^2}$. We should therefore add $\frac{b^2}{4a^2}$ to both sides and subtract $\frac{c}{a}$ from both sides to get:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

We can complete the square on the left hand side and subtract the fractions on the right hand side:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

We reach an important point. If the right hand side is less than 0, then the equation will have no real solution. The sign of the right hand side depends on the numerator only (as $4a^2$ is always positive). So the existence of real solutions of a quadratic equation depends on the expression $b^2 - 4ac$. This expression is called the **discriminant** of a quadratic and is denoted using the Greek letter Δ.

$$\Delta = b^2 - 4ac$$

If $\Delta = 0$, then the equation becomes:

$$\left(x + \frac{b}{2a}\right)^2 = 0$$

and we get only one solution, which is $x = -\frac{b}{2a}$. If $\Delta > 0$, then we get that:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

And finally we get, what is known as the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

Worked example 2.4.5

Solve the following equation using the quadratic formula:

$$2x^2 + 3x - 5 = 0$$

We have $a = 2$, $b = 3$ and $c = -5$. We will calculate the discriminant first:

$$\Delta = 3^2 - 4(2)(-5) = 49$$

Δ is positive, so there will be two real solutions. These are:

$$x = \frac{-3 \pm \sqrt{49}}{4} = \frac{-3 \pm 7}{4}$$

Which gives $x = 1$ or $x = -\frac{5}{2}$.

Exercise 2.4.5

Solve the following equations using the quadratic formula:

(a) $x^2 - 2x - 8 = 0$

(b) $x^2 + 4x - 5 = 0$

(c) $x^2 - 6x + 8 = 0$

(d) $x^2 + 5x + 6 = 0$

(e) $x^2 - 3x - 10 = 0$

(f) $x^2 + 9x + 18 = 0$

(g) $2x^2 + 8x + 5 = 0$

(h) $2x^2 + 4x - 6 = 0$

(i) $2x^2 + 12x + 18 = 0$

(j) $3x^2 - 4x - 5 = 0$

(k) $4x^2 + 2x - 8 = 0$

(l) $5x^2 - 6x + 1 = 0$

(m) $x^2 + 2x + 5 = 0$

(n) $6x^2 + 5x - 9 = 0$

(o) $x^2 + 4x + 4 = 0$

(p) $5x^2 + 3x - 7 = 0$

(q) $x^2 - 4x + 9 = 0$

(r) $2x^2 + x + 5 = 0$

(s) $x^2 - 5x - 6 = 0$

(t) $3x^2 + 3x - 4 = 0$

(u) $x^2 + 7x - 1 = 0$

(v) $2x^2 - 6x + 10 = 0$

(w) $x^2 + 3x - \frac{1}{2} = 0$

(x) $4x^2 - 5x + 2 = 0$

(y) $x^2 + 6x + \frac{2}{3} = 0$

(z) $5x^2 - \frac{1}{2}x + 3 = 0$

(aa) $2x^2 + \frac{1}{3}x - 5 = 0$

Exercise 2.4.6
Solve the following equations using any method:

(a) $x^2 - 4x = 0$

(b) $2x^2 + 3x = 0$

(c) $3x^2 = 8x$

(d) $x^2 - 9 = 0$

(e) $16x^2 - 25 = 0$

(f) $100x^2 = 49$

(g) $x^2 + 9 = 6x$

(h) $x^2 + 9x + 20 = 2$

(i) $x^2 = 10x - 21$

(j) $x^2 + 3x = 10$

(k) $x^2 = x + 12$

(l) $x^2 + 6x = 24 + x$

(m) $2x^2 + 5x = 8$

(n) $3x^2 = 5x + 2$

(o) $4x^2 + 4x = x + 7$

(p) $x^2 + 4x + 6 = 1$

(q) $x^2 + 2x + 6 = x - 1$

(r) $2x^2 - 3x - 5 = 0$

(s) $x^2 - 4x + 10 = 0$

(t) $5x^2 + 6x = 8$

(u) $2x^2 + 7x = 6$

(v) $3x^2 + 9 = 2x + 4$

(w) $x^2 + 12 = 5x + 2$

(x) $4x^2 = 8 - x$

(y) $2x^2 - 3x + 7 = -1$

(z) $x^2 + 3x + 11 = 12$

(aa) $3x^2 + 4x + 10 = 13$

(ab) $(3x-1)^2 - (x+2)^2 = 0$

(ac) $(x+1)^2 - (2x-3)^2 = 0$

(ad) $(2x+5)^2 = (x-1)^2$

(ae) $(5x-2)^2 = (3x-4)^2$

(af) $(x-2)^2 = (2x-3)^2 - 5$

(ag) $(x+1)^2 = (3x-1)^2 + 2$

(ah) $(x-3)^2 = (2x-1)^2 - 4x$

(ai) $(2x+3)^2 = (x-2)(x+2) + 4$

(aj)* $(x+1)^2 = \left(x^2 - 5\right)^2$

Worked example 2.4.7

Solve the following equation and state the necessary assumptions:

$$\frac{x+1}{x-2} = \frac{2x-1}{x+2}$$

We start by noting that $x \neq 2$ and $x \neq -2$ in order for the denominator not be 0. We can then multiply both sides by $x-2$ and $x+2$ to get:

$$x^2 + 3x + 2 = 2x^2 - 5x + 2$$

Which gives:

$$0 = x^2 - 8x$$

So $x(x-8) = 0$, which gives $x = 0$ or $x = 8$.

Exercise 2.4.7

Solve the following equations using any method (state necessary assumptions):

(a) $\dfrac{x+3}{x-1} = \dfrac{2x-1}{x+4}$

(b) $\dfrac{x}{x-2} = \dfrac{2x+1}{x-4}$

(c) $\dfrac{x-1}{2x+1} = \dfrac{x-3}{x-1}$

(d) $\dfrac{2x-3}{x+1} = \dfrac{x+3}{x+5}$

(e) $\dfrac{x-4}{x+1} = \dfrac{x}{2x-7}$

(f) $\dfrac{2x}{x-5} = \dfrac{x-2}{x+1}$

(g) $\dfrac{x-2}{x-1} = \dfrac{2x-1}{x+1}$

(h) $\dfrac{x+2}{x+3} = \dfrac{x+1}{2x-1}$

(i) $\dfrac{x}{x+3} = \dfrac{2x+1}{x-1}$

(j) $\dfrac{x-2}{x+1} = \dfrac{x-3}{x+3} - 1$

(k) $\dfrac{x}{x+3} = \dfrac{x+1}{x-1} + 1$

(l) $\dfrac{x-1}{x+2} = \dfrac{x+1}{x-1} - 2$

Imaginary number

1. Consider the equation:

$$x^2 + 1 = 0$$

Explain why this equation has no real solutions.

Assume that there is a number i, such that $i^2 = -1$. Note that i is not a real number, that is $i \notin \mathbb{R}$.

2. Write down the value of $(-i)^2$ and hence state the solutions to the equation:

$$x^2 + 1 = 0$$

3. Calculate the value of $i^3, \quad i^4, \quad i^{100}, \quad i^{123}$.

3. Calculate the value of $(2i)^2, \quad (-3i)^2, \quad (i\sqrt{3})^2, \quad (\frac{i\sqrt{6}}{2})^2$.

4. Write down the solutions to the following equations:

$$x^2 + 4 = 0, \qquad x^2 + 9 = 0, \qquad x^2 + 3 = 0, \qquad 2x^2 + 3 = 0$$

5. Expand and simplify the following expressions:

$$(1 + i)^2, \qquad (2 - i)^2, \qquad (3 + i)^2, \qquad (1 + 2i)^2$$

6. By completing the square, or otherwise, solve the following equations (your answers will involve i):

(a) $x^2 - 2x + 2 = 0$

(b) $x^2 - 4x + 5 = 0$

(c) $x^2 - 6x + 10 = 0$

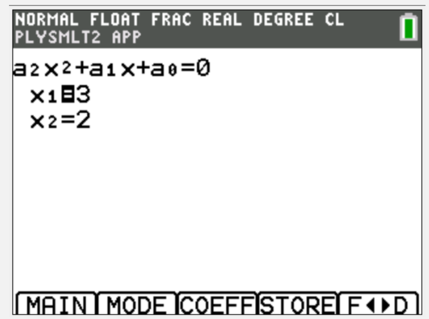
(d) $x^2 - 2x + 5 = 0$

(e) $x^2 + 6x + 12 = 0$

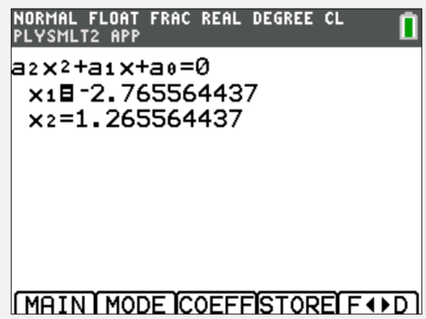
(f) $x^2 - 10x + 27 = 0$

Quadratic equations can be solved on GDC using the PlySmlt2 APP.

EXAMPLE 1

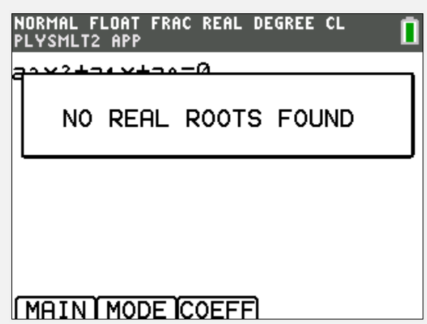
Solve:	**INPUT**	**OUTPUT**
$x^2-5x+6=0$		

EXAMPLE 2

Solve:	**INPUT**	**OUTPUT**
$2x^2+3x-7=0$		

If the solutions are irrational, the calculator will give you rounded answers as in the example above.

EXAMPLE 3

Solve:	**INPUT**	**OUTPUT**
$3x^2-2x+7=0$		

Note that in the last example we have $\Delta = -80 < 0$, so there are no real solutions to this equation.

SHORT TEST

1. [2 *points*]

Solve the following equations:

(a) $5x^2 = 4x$

(b) $9x^2 - 4 = 0$

2. [2 *points*]

Solve the following equations using factorization:

(a) $x^2 - 3x = 28$

(b) $2x^2 - x - 6 = 0$

3. [2 *points*]

Solve the following equations by completing the square:

(a) $x^2 - 4x = 45$

(b) $x^2 - 10x + 14 = 0$

4. [2 *points*]

Solve the following equations using quadratic formula:

(a) $2x^2 + 3x = 1$

(b) $x^2 + 3x + 7 = 0$

SHORT TEST

SOLUTIONS

1. [2 *points*]

Solve the following equations:

(a) $5x^2 = 4x$

$5x^2 - 4x = 0$

$x(5x - 4) = 0$

$x = 0$ or $x = \frac{4}{5}$

(b) $9x^2 - 4 = 0$

$(3x - 2)(3x + 2) = 0$

$x = \frac{2}{3}$ or $x = -\frac{2}{3}$

2. [2 *points*]

Solve the following equations using factorization:

(a) $x^2 - 3x = 28$

$x^2 - 3x - 28 = 0$

$(x + 4)(x - 7) = 0$

$x = -4$ or $x = 7$

(b) $2x^2 - x - 6 = 0$

$(2x + 3)(x - 2) = 0$

$x = -\frac{3}{2}$ or $x = 2$

3. [2 *points*]

Solve the following equations by completing the square:

(a) $x^2 - 4x = 45$

$x^2 - 4x + 4 = 49$

$(x - 2)^2 = 49$

$x - 2 = \pm 7$

$x = -5$ or $x = 9$

(b) $x^2 - 10x + 14 = 0$

$x^2 - 10x + 25 = 11$

$(x - 5)^2 = 11$

$x - 5 = \pm\sqrt{11}$

$x = 5 - \sqrt{11}$ or $x = 5 + \sqrt{11}$

4. [2 *points*]

Solve the following equations using quadratic formula:

(a) $2x^2 + 3x = 11$

$a = 2,\ b = 3,\ c = -11$

$\Delta = 3^2 - 4(2)(-11) = 97$

$$x = \frac{-3 \pm \sqrt{97}}{4}$$

(b) $x^2 + 3x + 7 = 0$

$a = 1,\ b = 3,\ c = 7$

$\Delta = 3^2 - 4(1)(7) = -19 < 0$

no real solutions

1.5 Inequalities

Consider the following inequality:

$$3x - 8 \geq 4$$

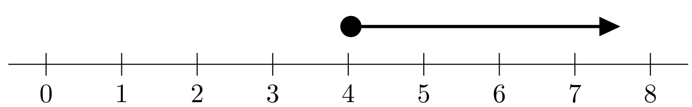
7 satisfies this inequality, because $3 \times 7 - 8 = 13$ and $13 \geq 4$. Similarly 4 satisfies this inequality, because $3 \times 4 - 8 = 4$ and $4 \geq 4$. However 1 does not satisfy this inequality because $3 \times 1 - 8 = -5$ and $-5 < 4$ (so it is not true that $-5 \geq 4$). To find all values of x which satisfy this inequality we rearrange it by adding 8 to both sides:

$$3x \geq 12$$

and dividing both sides by 3:

$$x \geq 4$$

Now we know that the inequality is satisfied by every real number which is greater or equal to 4. This can be represented on the number line as follows:



We can also write the solutions to the above inequality using interval notation as $x \in [4, \infty[$.

Now consider the following inequality:

$$15 - 7x > 1$$

We subtract 15 from both sides to get:

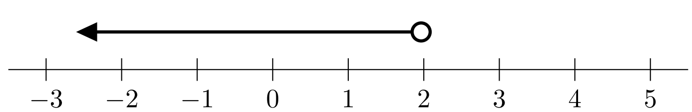
$$-7x > -14$$

Now we divide both sides by -7 and get:

$$x < 2$$

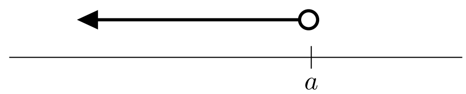
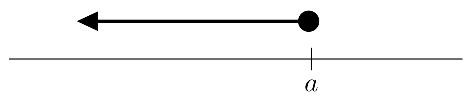
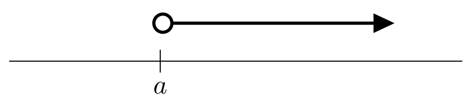
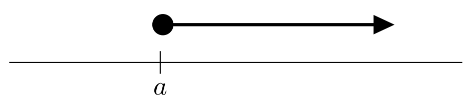
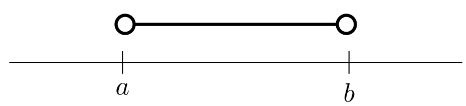
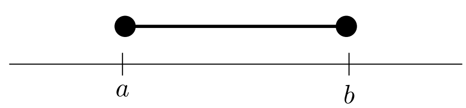
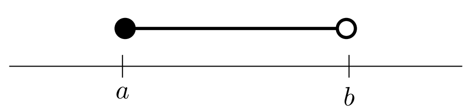
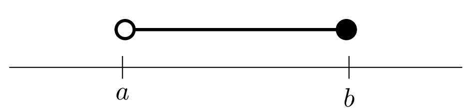
Note that the inequality is reversed. **Multiplying or dividing both sides of an inequality by a negative number reverses the inequality.**

The solutions to the inequality can be represented on the number line as:



Using the interval notation we write this as $x \in]-\infty, 2[$.

We use the following notation:

Inequality	Number line	Interval
$x < a$		$]-\infty, a[$
$x \leq a$		$]-\infty, a]$
$x > a$		$]a, \infty[$
$x \geq a$		$[a, \infty[$
$a < x < b$		$]a, b[$
$a \leq x \leq b$		$[a, b]$
$a \leq x < b$		$[a, b[$
$a < x \leq b$		$]a, b]$

LINEAR INEQUALITIES

Worked example 2.5.1

Consider the following inequality:

$$11 - 3x < 4$$

(a) Decide if the following numbers satisfy the inequality:

(i) 1 (ii) $\sqrt{5}$ (iii) π

(b) Solve the inequality, write your answer using interval notation and represent the solution on the number line.

(a) (i)

$$11 - 3 \times 1 = 8 > 4$$

So 1 does not satisfy the inequality.

(ii) We need to decide whether $11 - 3\sqrt{5}$ is greater than 4 or not. Note that $\sqrt{5}$ is between 2 and 3, but this is not enough to answer the question as $11 - 3 \times 2 = 5 > 4$, but $11 - 3 \times 3 = 2 < 4$. However $3\sqrt{5} = \sqrt{45}$ and $\sqrt{45} < \sqrt{49} = 7$, so we are subtracting a number smaller than 7, which means that $11 - 3\sqrt{5} > 4$ i.e. $\sqrt{5}$ does not satisfy the inequality.

(iii) $\pi > 3$, so $11 - 3\pi < 11 - 3 \times 3 = 2 < 4$, so $11 - 3\pi < 4$ i.e. π satisfies the inequality.

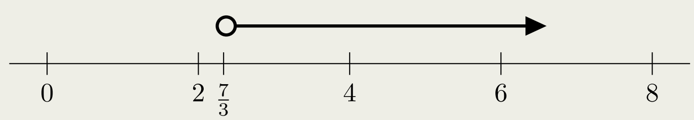
(b) We subtract 11 from both sides:

$$-3x < -7$$

and divide both sides by -3 (remembering to reverse the inequality):

$$x > \frac{7}{3}$$

So $x \in]\frac{7}{3}, \infty[$ and this can be represented on the number line as:



Exercise 2.5.1 Decide if the following numbers satisfy the given inequality:

(a) $3,\ \sqrt{10},\ \pi,\quad 4x - 5 < 9$

(b) $0,\ \sqrt[3]{2},\ 1.\dot{2},\quad 3x + 2 \leq 6$

(c) $1,\ \sqrt{2},\ \sqrt[3]{2},\quad 5 - 6x < -2$

(d) $1.\dot{6},\ \sqrt[3]{3},\ 2^{-1},\quad 7 - 3x \geq 2$

(e) $\sqrt{2},\ \sqrt{3},\ 2,\quad x\sqrt{2} - \sqrt{3} < 1$

(f) $2,\ 2^2,\ 2^{-2},\quad 2^5x + 2^6 \leq 2^7$

Exercise 2.5.2a Solve the following inequalities, write your answer using interval notation and represent it on a number line:

(a) $3x - 5 < 1$

(b) $4 - 3x \leq 2$

(c) $7 - 2x > 2$

(d) $4 - 5x \geq 12$

(e) $5 < 3 - 10x$

(f) $7 \geq 4 + 3x$

(g) $2x + 3 < 3x - 4$

(h) $7 - x \geq 3 + 2x$

(i) $11 - 5x < 3 + 4x$

(j) $\frac{3}{4}x - \frac{1}{3} \geq \frac{5}{6}$

(k) $\frac{1}{5}x - \frac{1}{2} < \frac{2}{3}$

(l) $3x - 4 > \frac{1}{4}x - \frac{1}{5}$

(m) $\frac{x-5}{3} + \frac{x}{4} \leq 2$

(n) $\frac{1-2x}{7} + \frac{1}{3} > \frac{x+2}{4}$

(o) $\frac{x}{4} - \frac{x+1}{2} < \frac{2x-5}{3}$

(p) $7x - 3 \leq 7x + 5$

(q) $4x - 5 \geq 2(2x - 2)$

(r) $3(x - 1) + 2 < 3x - \frac{1}{4}$

(s) $x\sqrt{2} + 1 > 3$

(t) $x\sqrt{3} - 3\sqrt{3} < 5$

(u) $2x\sqrt{5} - \sqrt{3} \leq \sqrt{2}$

(v) $x\sqrt{2} + 3 > x + 4$

(w) $x\sqrt{3} - 5 < 3 - x$

(x) $2x\sqrt{2} - \sqrt{5} \geq 4x$

(y) $3x + 1 \geq x\sqrt{3} - 1$

(z) $x\sqrt{2} - 1 < x\sqrt{3} + 2$

(zz) $\frac{1}{2}x - 3 \leq x\sqrt{2} + 5$

Exercise 2.5.2b Solve the following inequalities, write your answer using interval notation and represent it on a number line:

(a) $(x + 2)^2 - 3x > (x - 1)(x + 1)$

(b) $(x - 1)^2 + 4x > (x - 2)(x + 2) + 5$

(a) $2x^2 - (x + 1)^2 > x(x - 3) + 2$

(b) $(x + 1)^2 + (x - 2)^2 > 2(x - 3)(x + 2)$

(a) $(2x + 1)^2 - 3x(x + 1) > (x - 2)^2$

(b) $2x(x - 1) - (x + 4)^2 > (x - 3)(x + 3)$

(a) $(x + 2)^2 - 3x > (x - 1)(x + 1)$

(b) $(x + 2)^2 - 3x > (x - 1)(x + 1)$

(a) $(x + \sqrt{2})^2 + x > (x - \sqrt{3})(x + \sqrt{3})$

(b) $(x - \sqrt{3})^2 + 2x > (x - \sqrt{2})(x + \sqrt{2})$

(a) $(x + \frac{1}{2})^2 - \frac{1}{3}x > \frac{1}{4}(2x - 3)(2x + 3)$

(b) $2(x - \frac{1}{2})^2 + 2x(x + 1) > (2x - \frac{1}{2})^2$

QUADRATIC INEQUALITIES

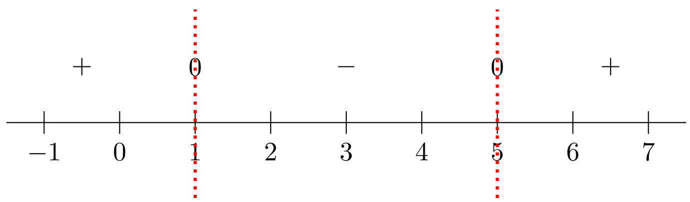
Consider the following inequality:

$$x^2 - 6x + 5 > 0$$

One side of this inequality is a quadratic expression, while the other side is 0. We can solve this inequality by factoring the quadratic expression:

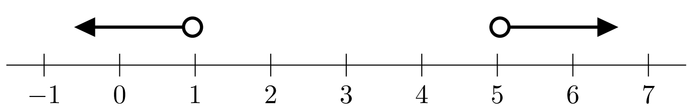
$$(x-1)(x-5) > 0$$

We can analyze the sign of the factored expression. When $x = 1$ or $x = 5$, the expression is 0. When $x > 5$, then both $x - 1$ and $x - 5$ are positive, so their product is positive. When $x < 1$, both $x - 1$ and $x - 5$ are negative, so their product is again positive. Finally if $1 < x < 5$, then $x - 1$ is positive, but $x - 5$ is negative, so their product is negative. This can be represented on a **sign diagram**:



Note that +, − and 0 above the number line represent the sign of the expression for the given interval.

Because we want $(x-1)(x-5)$ to be greater than 0, then the solutions are $x < 1$ or $x > 5$. This can be represented on the number line as:



Using interval notation we can write $x \in]-\infty, 1[\cup]5, \infty[$.

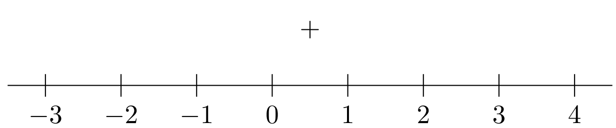
Now consider the following inequality:

$$x^2 + 2x + 7 > 0$$

We cannot factorize the expression on the left hand side (at least not using real numbers). In fact we get that $\Delta = 4 - 28 = -24 < 0$, which means that the quadratic expression is never 0. We can rewrite it by completing the square to get:

$$(x+1)^2 + 6 > 0$$

Now we can clearly see that the left hand side is always positive as $(x+1)^2$ is nonnegative and we add 6 to it. This means that the inequality is satisfied for all real numbers x ($x \in \mathbb{R}$). In fact the sign diagram for $x^2 + 2x + 7$ is simply:



Indicating that the expression is positive for all real numbers.

Worked example 2.5.3

Solve the following inequality:

$$x^2 + 4x \leq 1$$

We start by moving all terms to one side:

$$x^2 + 4x - 1 \leq 0$$

Now we would like to factorize the quadratic expression on the left hand side. We can proceed in two ways.

Method 1
We complete the square and rewrite the *LHS* as follows:

$$(x + 2)^2 - 5 \leq 0$$

Now we use difference of squares to factorize the *LHS*:

$$(x + 2 - \sqrt{5})(x + 2 + \sqrt{5}) \leq 0$$

Method 2 We use the quadratic formula to find the zeroes of $x^2 + 4x - 1$ and use the fact that we have:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

where x_1 and x_2 are the zeroes of $ax^2 + bx + c$.

$$x_{1,2} = \frac{-4 \pm \sqrt{16 + 4}}{2} = -2 \pm \sqrt{5}$$

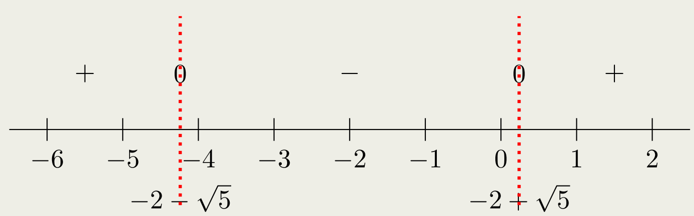
and since the coefficient of x^2 is just 1, we must have:

$$x^2 + 4x - 1 = \left(x - (-2 + \sqrt{5})\right)\left(x - (-2 - \sqrt{5})\right)$$

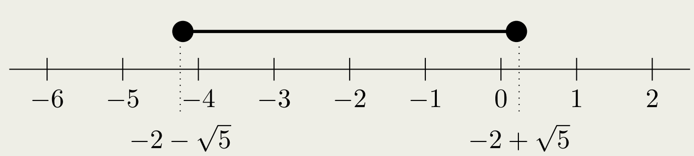
so:

$$x^2 + 4x - 1 = (x + 2 - \sqrt{5})(x + 2 + \sqrt{5})$$

Once we factorized the quadratic expression using either of the two methods we analyze the sign of the expression and draw the sign diagram:



We want our expression to be smaller or equal to zero so the solution is $-2 - \sqrt{5} \leq x \leq -2 + \sqrt{5}$. On the number line this can be represented as:



Using interval notation we have $x \in [-2 - \sqrt{5}, -2 + \sqrt{5}]$.

Exercise 2.5.3 Solve the following inequalities. Write your answer using interval notation and represent it on the number line:

(a) $x^2+5x<0$

(b) $2x^2-7x\geq 0$

(c) $3x^2-x>0$

(d) $5x^2>x$

(e) $6x^2\leq 7x$

(f) $2x^2>3x$

(g) $x^2-4\leq 0$

(h) $9x^2-16>0$

(i) $x^2-3<0$

(j) $4x^2>9$

(k) $100x^2\geq 1$

(l) $x^2<11$

(m) $3x^2>2$

(n) $5x^2\geq 8$

(o) $2x^2<\frac{1}{3}$

(p) $x^2-x-2<0$

(q) $x^2-3x\geq 10$

(r) $x^2+16>10x$

(s) $x^2\leq 7x-12$

(t) $x+20<x^2$

(u) $x^2+x\geq 3x+15$

(v) $5x<3-2x^2$

(w) $2x^2-x\geq 6$

(x) $2x^2<x+10$

(y) $3x^2+5x\geq 4+x$

(z) $3x^2<x+4$

(aa) $3x^2+x>5x+15$

(ab) $x^2+6x+7\leq 0$

(ac) $x^2-4x>1$

(ad) $10x<x^2+19$

(ae) $2x^2\leq 3x+1$

(af) $5x\geq x^2-2$

(ag) $1-x^2<3x$

(ah) $x^2+5x+8>0$

(ai) $x^2+4\leq 4x$

(aj) $0\leq x^2+6x+9$

(ak) $x^2+5<x$

(al) $x^2\leq 8x-16$

(am) $2x^2\geq x-2$

(an) $6x-8\geq x^2$

(ao) $7x+5<2x^2$

(ap) $x^2\leq 2x-1$

(aq) $x^2-x\sqrt{2}<x-\sqrt{2}$

(ar) $x^2+x\sqrt{3}>2x+\sqrt{12}$

(as) $x^2\leq x\sqrt{3}-x\sqrt{2}+\sqrt{6}$

(at) $2x^2+4x>x\sqrt{5}+\sqrt{20}$

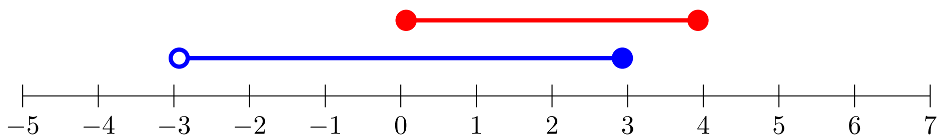
(au) $3x^2+x\sqrt{18}\geq x+\sqrt{2}$

(av) $4x^2+x\sqrt{48}<3x+\sqrt{27}$

WORKING WITH INTERVALS

An interval is a set, so we can perform set operations like *union*, *intersection*, *difference* and *complement* on intervals. In this section we will revisit set operations of $\cup$ $\cap$ $-$ and ^{c}.

Consider the intervals $A =]-3, 3]$ and $B = [0, 4]$. We can represent each of these intervals on a number line (A in blue, B in red):



Now we can see that some numbers (for example 1.5, $\sqrt{5}$ and 3) are in both A and B, in fact every number between 0 and 3 (inclusive) is in both A and B, so we can write that:

$$A \cap B = [0, 3]$$

Every number between -3 (exclusive) and 4 (inclusive) is in A or B, so:

$$A \cup B =]-3, 4]$$

Every number between -3 (exclusive) and 0 (exclusive) is in A, but not in B, so:

$$A - B =]-3, 0[$$

Every number between 3 (exclusive) and 4 (inclusive) is in B, but not in A, so:

$$B - A =]3, 4]$$

It is important to realize that although $B - A$ contains only one integer (4), it has infinitely many elements that are not integers (for example 3.5, $\sqrt{10}$ and π).

Numbers not in A are those that are smaller or equal to -3 and those that are greater than 3, so:

$$A^c =]-\infty, -3] \cup]3, \infty[$$

Similarly numbers not in B are those that are smaller than 0 and those that are greater than 4:

$$B^c =]-\infty, 0[\cup]4, \infty[$$

Note that the universal set U has not been specified, so we assume the universal set to be all real numbers, that is $U = \mathbb{R}$.

Worked example 2.5.4

Given the following intervals:

$A = [-2, 3]$

$B =]-\infty, 1]$

$C =]0, \infty[$

Write down the following:

(a) $A \cap B$

(b) $B \cap C$

(c) $A - B$

(d) $A - C$

(e) $A \cup B$

(f) $B \cup C$

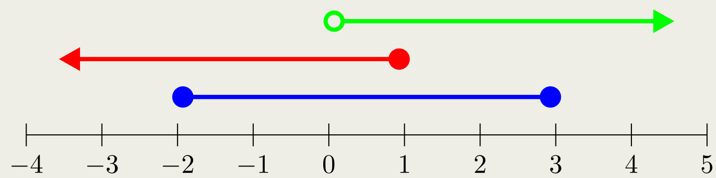
(g) A^c

(h) B^c

(i) C^c

(j) $A \cap B \cap C$

The three intervals can be represented on a number line (A - blue; B - red; C - green):



(a) $A \cap B$ is the part that is both blue and red, so we have

$$A \cap B = [-2, 1]$$

(b) $B \cap C$ is the part that is both red and yellow, so we have

$$B \cap C =]0, 1]$$

(c) $A - B$ is the part that is blue but not red, so we have

$$A - B =]1, 3]$$

Note that 1 is not in $A - B$ as it is in B.

(d) $A - C$ is the part that is blue but not yellow, so we have

$$A - C = [-2, 0]$$

Note that 0 is in $A - C$ as it is in A and is not in C.

(e) $A \cup B$ is the part that is blue or red, so we have

$$A \cup B =]-\infty, 3]$$

(f) $B \cup C$ is the part that is red or yellow, so we have

$$B \cup C = \mathbb{R}$$

as every number on the number line is in B or C.

(g) A^c is the part that is not blue, so we have

$$A^c =]-\infty, -2[\cup]3, \infty[$$

(h) B^c is the part that is not red, so we have

$$B^c =]1, \infty[$$

(i) C^c is the part that is not yellow, so we have

$$C^c =]-\infty, 0]$$

(j) $A \cap B \cap C$ is the part that is both blue, red and yellow, so

$$A \cap B \cap C =]0, 1]$$

Exercise 2.5.4a
Find $A \cap B$, $A \cup B$, $A - B$, $B - A$, A^c and B^c for the following intervals:

(a) $A =]-\infty, 3]$ $\quad B = [-1, 4]$

(b) $A =]-1, 5[$ $\quad B = [0, \infty[$

(c) $A = [-2, 1]$ $\quad B = [0, 3[$

(d) $A = [-3, 1[$ $\quad B =]-2, 5[$

(e) $A =]-\infty, 1]$ $\quad B =]0, \infty]$

(f) $A =]-2, \infty[$ $\quad B = [-\infty, 3[$

(g) $A =]-3, 2]$ $\quad B =]-2, 2[$

(h) $A =]-4, 4[$ $\quad B = [0, 1]$

Exercise 2.5.4b
Given that:

$$A =]-\infty, 2] \qquad B =]-3, \infty[\qquad C = [-1, 4] \qquad D =]-2, 2[$$

Find:

(a) $A \cap B$ (b) $A \cap C$ (c) $B \cap C$ (d) $C \cap D$

(a) $A \cup B$ (b) $A \cup C$ (c) $B \cup C$ (d) $C \cap D$

(a) $A - B$ (b) $B - A$ (c) $A - C$ (d) $C - A$

(a) $C - D$ (b) $D - C$ (c) $B - D$ (d) $D - B$

(a) A^c (b) B^c (c) C^c (d) D^c

(a) $A \cap B \cap C$ (b) $(A \cap C) - B$ (c) $(A \cap B) - D$ (d) $C - (A \cap C)$

(a) $A \cap \mathbb{N}$ (b) $C \cap \mathbb{Z}$ (c) $D \cap \mathbb{N}$ (d) $A \cap B \cap \mathbb{Z}$

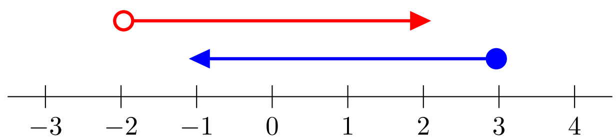
(a) $A^c \cap C \cap \mathbb{N}$ (b) $(C - D) \cap \mathbb{Z}$ (c) $B^c \cap \mathbb{N}$ (d) $C \cap D \cap \mathbb{Z}$

SYSTEMS OF INEQUALITIES

Consider the following system of inequalities:

$$\begin{cases} 2x - 5 \leq 1 \\ 1 - 3x < 7 \end{cases}$$

The first inequality is satisfied by $x \leq 3$ (blue) and the second inequality is satisfied $x > -2$ (red):



so both inequalities are simultaneously satisfied by $-2 < x \leq 3$ (both blue and red) or $x \in]-2, 3]$.

Worked example 2.5.5

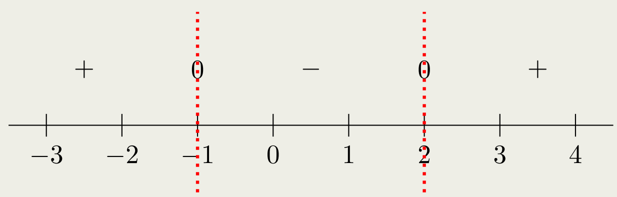
Solve the following system of inequalities:

$$\begin{cases} 1 - 2x < 5 \\ x^2 \geq x + 2 \end{cases}$$

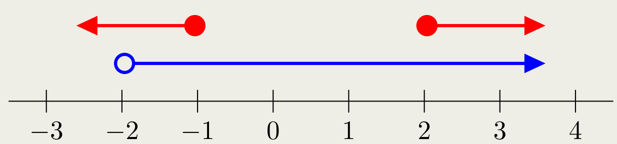
We solve both inequalities:

$$\begin{cases} -2x < 4 \\ x^2 - x - 2 \geq 0 \end{cases} \qquad \begin{cases} x > -2 \\ (x+1)(x-2) \geq 0 \end{cases}$$

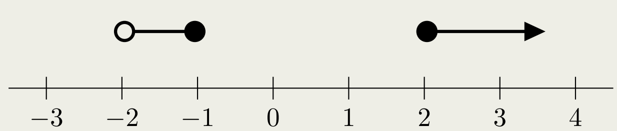
We draw the sign diagram for the second inequality:



Now we can indicate solutions to both inequalities on the number line:



Blue arrow represents the first inequality, red arrows represent the second. Now the solution to the system is:



Which is $-2 < x \leq 1$ or $x \geq 2$.

Using interval notation $x \in]-2, -1] \cup [2, \infty[$.

Exercise 2.5.5 Solve the following systems of inequalities:

(a) $\begin{cases} 3x-1 \leq 5 \\ 2-5x < 12 \end{cases}$

(b) $\begin{cases} \dfrac{x-1}{3} > 1 \\ \dfrac{2-x}{4} \leq 1 \end{cases}$

(c) $\begin{cases} \dfrac{x}{2} - \dfrac{x-1}{3} \leq 2 \\ 3 \geq \dfrac{3-x}{2} \end{cases}$

(d) $\begin{cases} \dfrac{2x-1}{3} < 5 \\ \dfrac{x+1}{4} - \dfrac{x}{5} < 1 \end{cases}$

(e) $\begin{cases} 4x-7 \geq -1 \\ \dfrac{1-x}{2} < 3 \end{cases}$

(f) $\begin{cases} (x-1)^2 - 3 \leq (x+2)^2 \\ \dfrac{3x-1}{2} \leq 4 \end{cases}$

(g) $\begin{cases} \dfrac{4x-1}{3} \leq 5 \\ \dfrac{x-5}{2} + \dfrac{x}{3} > 5 \end{cases}$

(h) $\begin{cases} \dfrac{2x-3}{4} + 1 > \dfrac{x}{3} \\ \dfrac{4-3x}{2} - 7 \geq x \end{cases}$

(i) $\begin{cases} x^2 - 2x \leq 15 \\ \dfrac{5x-1}{2} < 7 \end{cases}$

(j) $\begin{cases} x^2 + 6 > 5x \\ (x-1)^2 \leq (x-2)(x+2) \end{cases}$

(k) $\begin{cases} x^2 > 4 \\ \dfrac{x-3}{3} - \dfrac{x+1}{4} < \dfrac{x}{2} \end{cases}$

(l) $\begin{cases} x^2 < 25 \\ x^2 + 12 \geq 7x \end{cases}$

(m) $\begin{cases} 2x \geq x^2 \\ \dfrac{x+4}{5} - \dfrac{x+2}{3} < \dfrac{x-1}{2} \end{cases}$

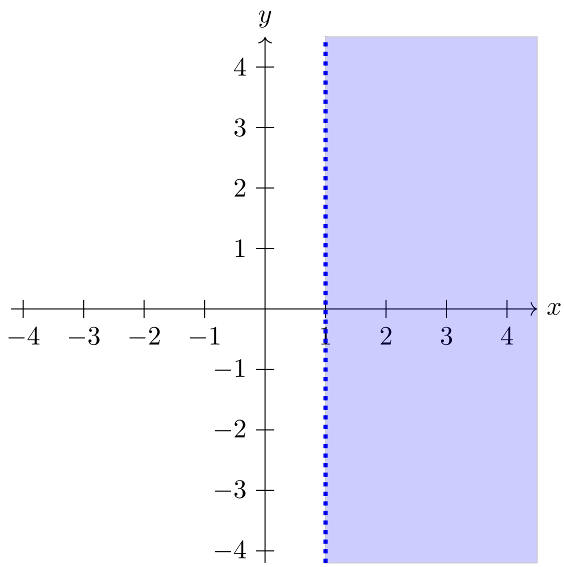
(n) $\begin{cases} x^2 < 10x \\ \dfrac{1-x}{3} \leq \dfrac{3-x}{5} \end{cases}$

(o) $\begin{cases} 7x-6 > x^2 \\ \dfrac{x-1}{2} + \dfrac{x-2}{3} + \dfrac{x-5}{4} < 3 \end{cases}$

(l) $\begin{cases} x^2 + 3x < 0 \\ x^2 + 13 \geq 5x \end{cases}$

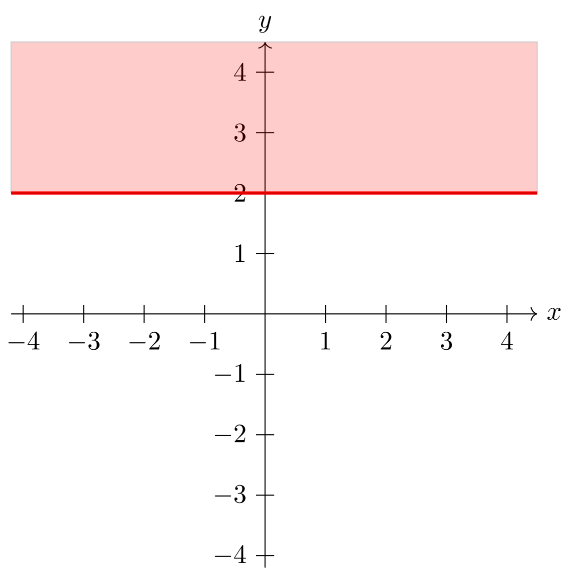
INEQUALITIES IN 2 DIMENSIONS

The inequalities can also be represented on a graph. Consider the inequality $x > 1$. In a coordinate system, where the first coordinate is x and the second coordinate is y, this inequality represents all points for which the first coordinate is greater than 1. This can be represented graphically as:



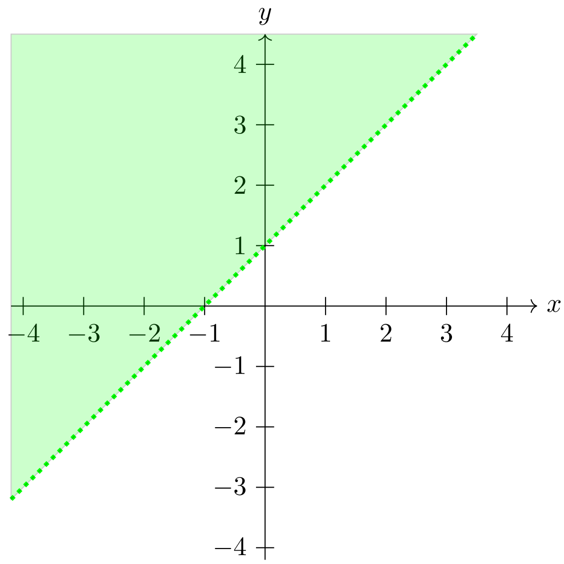
Note that the dotted line represents the fact that points with the first coordinate equal to 1 are not included. We can see that for example points $(3, 1)$ and $(2, -4)$ satisfy the inequality, while points $(-2, 3)$ and $(1, -2)$ do not.

Similarly the inequality $y \geq 2$ represents all points whose second coordinate is smaller or equal to 2:



Note that the thick line indicates the fact that points with second coordinate equal to 2 satisfy the inequality. Example of points satisfying the inequality: $(-3, \pi)$, $(1, 2)$ and $(\sqrt{2}, 3)$. Example of points not satisfying the inequality: $(3, 1.9)$, $(4, -4)$ and $(0, 0)$.

Now consider the inequality $y > x + 1$. Points whose second coordinate is more than 1 more than the first coordinate satisfy this inequality. We can represent this as:

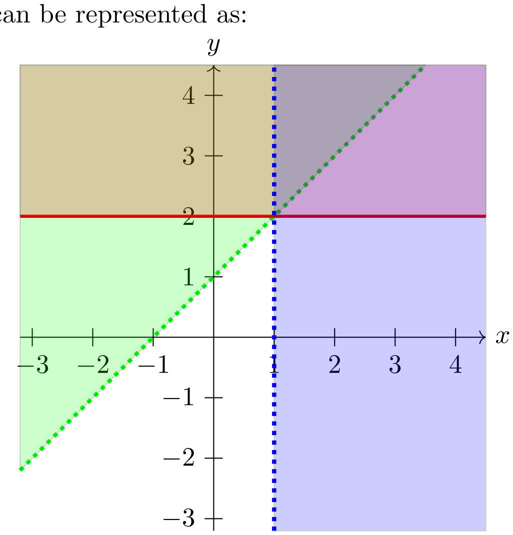


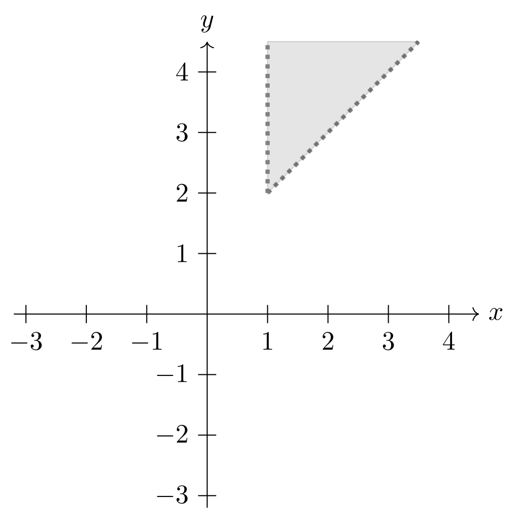
Note that the dotted line indicates the points where the second coordinate is equal to 1 plus the first coordinate - we do not include these points.

The set of points satisfying all of 3 of the above inequalities, that is:

$$\begin{cases} x > 1 \\ y \geq 2 \\ y > x + 1 \end{cases}$$

can be represented as:





The diagram on the left shows the regions satisfying each inequality: $x > 1$ in blue, $y \geq 2$ in red and $y > x+1$ in green. The diagram on the right shows the region satisfying all three inequalities. It corresponds to the region on the right diagram that is shaded with all three colours.

Exercise 2.5.6 Mark points satisfying the following inequalities on the coordinate system:

(a) $x < 2$

(b) $y \geq -1$

(c) $x \geq \frac{1}{2}$

(d) $x \leq -1$

(e) $y < \sqrt{2}$

(f) $2y + 3 > 0$

(g) $2x - 3 \geq 5$

(h) $3y + 7 \leq 10$

(i) $\frac{2x-3}{2} < 1$

(j) $y < x - 1$

(k) $x > y + 2$

(l) $y \geq 2x$

(m) $x + y < 3$

(n) $y - x \leq 2$

(o) $x \geq 2y$

Exercise 2.5.7 Mark points satisfying the following system of inequalities on the coordinate system:

(a) $\begin{cases} x > 0 \\ y \leq 1 \end{cases}$

(b) $\begin{cases} x \geq -1 \\ y < 3 \end{cases}$

(c) $\begin{cases} x \leq 3 \\ y < -1 \end{cases}$

(d) $\begin{cases} x > 1 \\ y \geq x \end{cases}$

(e) $\begin{cases} y \leq x \\ y < 2 \end{cases}$

(f) $\begin{cases} x > y - 1 \\ y \leq 2 \end{cases}$

(g) $\begin{cases} x + y > 0 \\ y \geq 3 \end{cases}$

(h) $\begin{cases} y > x + 1 \\ y \leq 1 - x \end{cases}$

(i) $\begin{cases} x > 2y \\ y < x + 1 \end{cases}$

(j) $\begin{cases} x > 1 \\ y \leq 1 \\ y \geq -2 \end{cases}$

(k) $\begin{cases} x < 3 \\ x \geq -1 \\ y \leq 3 \end{cases}$

(l) $\begin{cases} x > 0 \\ y \leq 3 \\ x \leq 4 \end{cases}$

(m) $\begin{cases} x > -1 \\ y \leq 2 \\ y < x \end{cases}$

(n) $\begin{cases} x \leq 3 \\ y \leq 2 \\ y < x + 1 \end{cases}$

(o) $\begin{cases} x \geq -1 \\ y \leq 4 \\ y > 2x \end{cases}$

(p) $\begin{cases} x \geq -2 \\ y \leq x \\ y < 2 - x \end{cases}$

(q) $\begin{cases} x > 0 \\ x + y \leq 4 \\ y > x - 2 \end{cases}$

(r) $\begin{cases} x + y \geq -3 \\ x + y < 1 \\ y > x + 1 \end{cases}$

Exercise 2.5.8 Choose all the points from the list below that satisfy the given inequality or system of inequalities.

$$A(1,1),\ B(1,-2),\ C(-2,2),\ D(-1,-3),\ E(\sqrt{2},1),\ F(\pi,\pi),\ G(\sqrt{5},\sqrt{10}),\ H(0.\dot{3},1-\sqrt{2})$$

(a) $x \geq 2$

(b) $y \leq 1$

(c) $y < x$

(d) $y + x \leq 2$

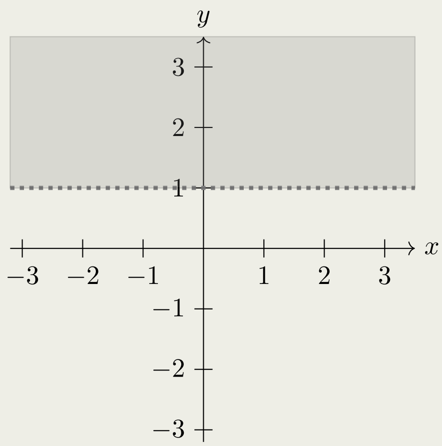
(e) $\begin{cases} x \geq 0 \\ y \leq 1 \end{cases}$

(f) $\begin{cases} x < 2 \\ y \leq 4 \end{cases}$

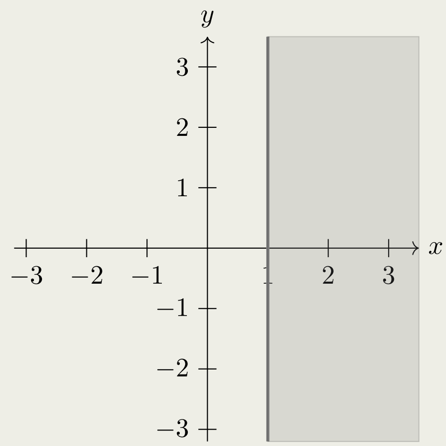
(g) $\begin{cases} y \geq x \\ y < 2 \end{cases}$

(h) $\begin{cases} x + y > 1 \\ y \geq 0 \end{cases}$

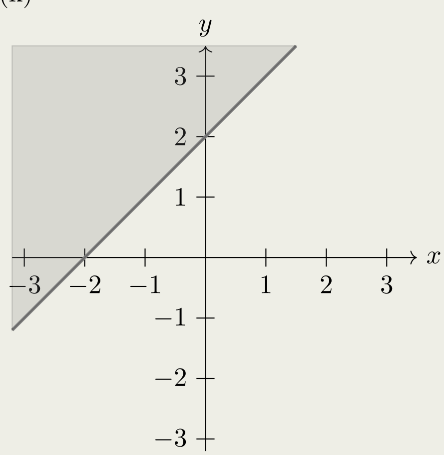
(i)



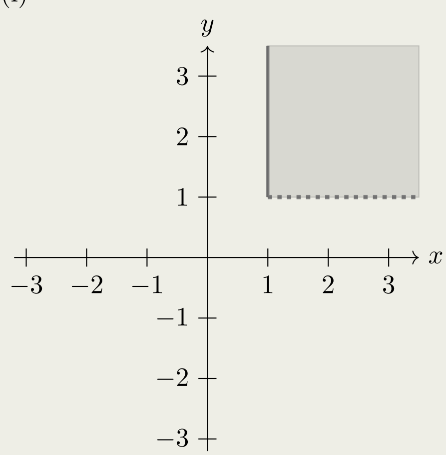
(j)



(k)

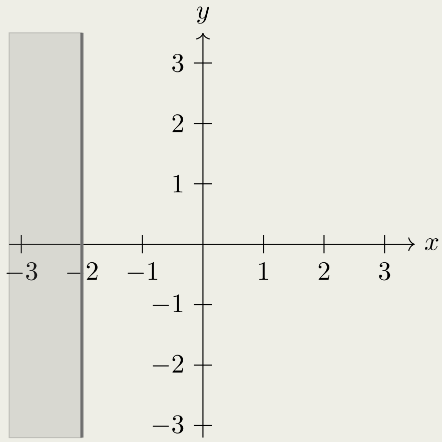


(l)

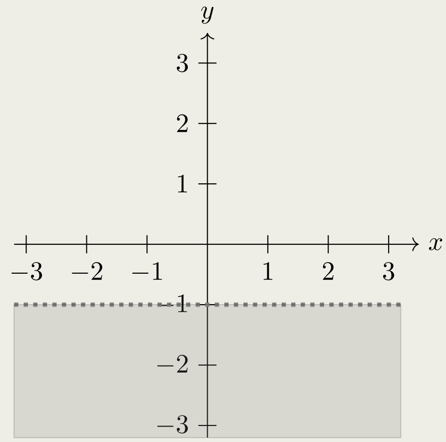


Exercise 2.5.9 Write down inequalities that correspond to the shaded regions.

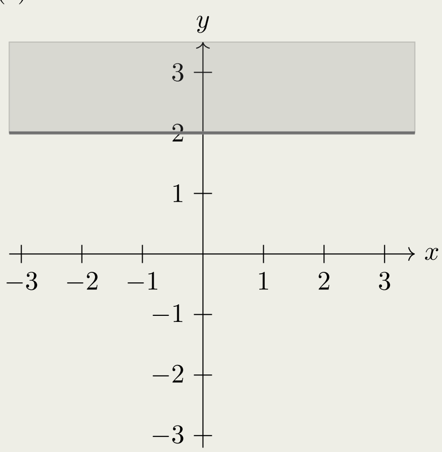
(a)



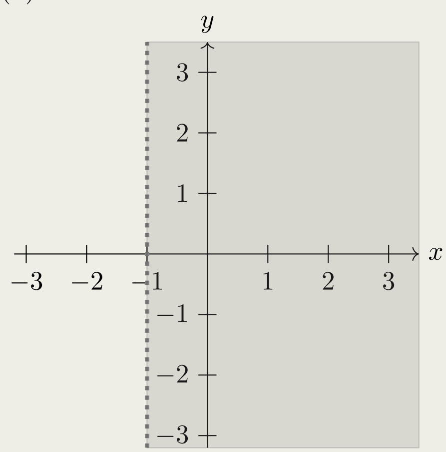
(b)



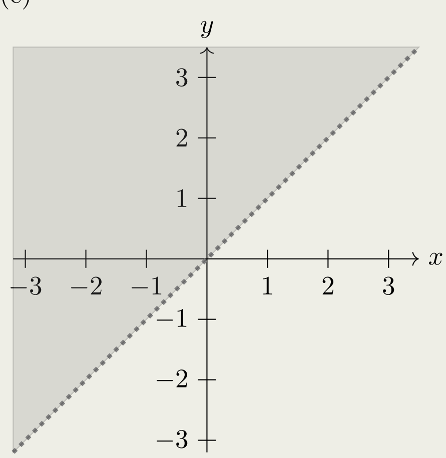
(c)



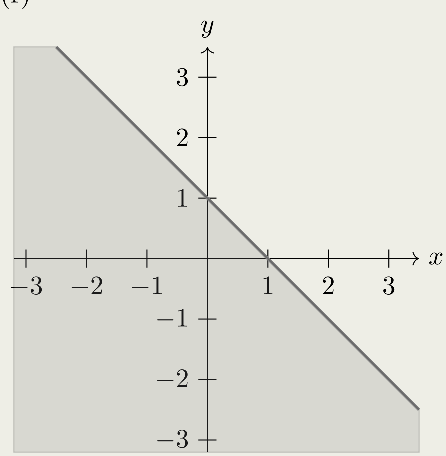
(d)



(e)

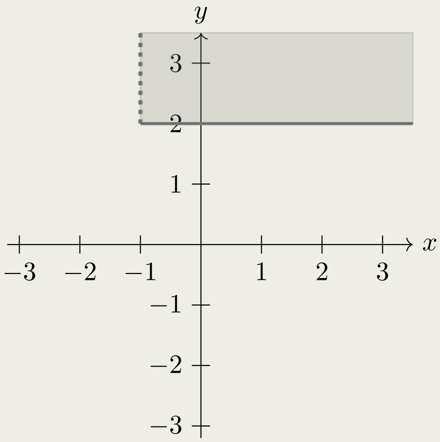


(f)

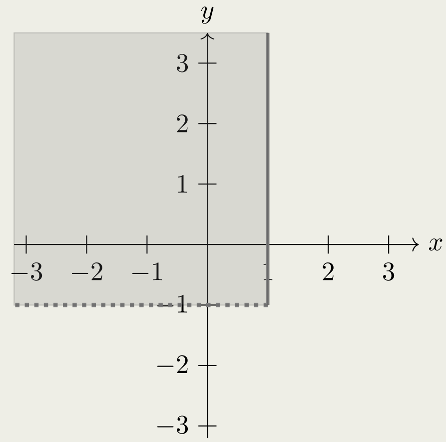


Exercise 2.5.2b Write down systems of inequalities that correspond to the shaded regions.

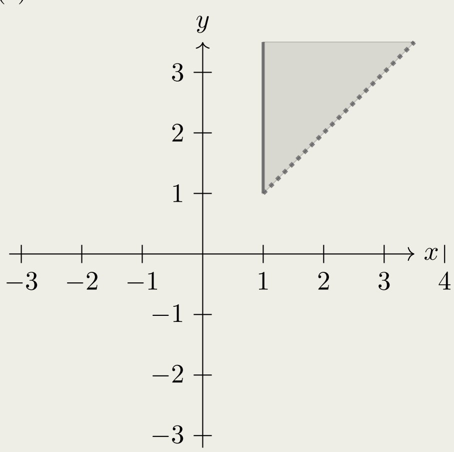
(a)



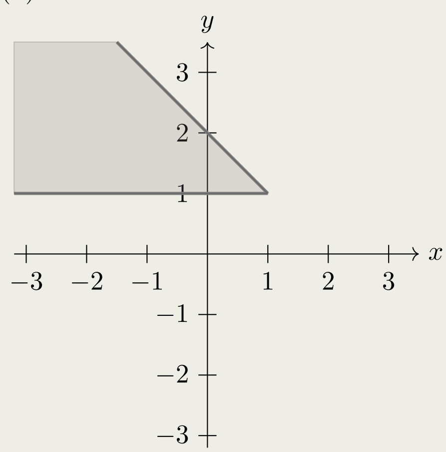
(b)



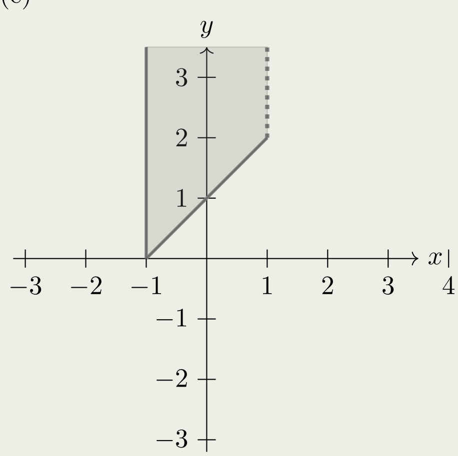
(c)



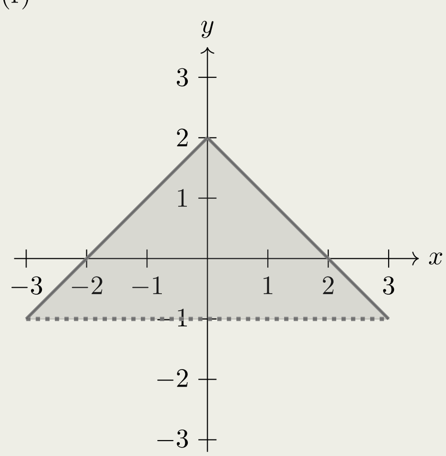
(d)



(e)



(f)



Further inequalities

1. Consider the expression $(x+2)(x-1)(x-3)$.

   (a) For what values of x is the expression 0?

   (b) Consider the sign of the expression for $x<-2$, $-2<x<1$, $1<x<3$ and $x>3$ and hence draw the sign diagram for this expression.

   (c) Solve the inequality:

$$(x+2)(x-1)(x-3)\geq 0$$

2. By drawing the sign diagram for the left hand side solve the following inequalities:

   (a) $x(x+3)(x-5)\leq 0$

   (b) $(x+4)(x+1)(x-3)(x-4)>0$

   (c) $x(x+3)^2(x-5)\leq 0$

   (d) $(x+4)^2(x+1)(x-3)^2(x-4)>0$

   (e) $x^3(x+3)^2(x-5)^2\leq 0$

   (f) $(x+4)^3(x+1)^4(x-3)^2(x-4)^2>0$

3. Consider the expression $\dfrac{x+3}{x-1}$.

   (a) For what value of x is the expression not defined?

   (b) For what value of x is the expression 0?

   (c) Consider the sign of the expression for $x<-3$, $-3<x<1$ and $x>1$ and hence draw the sign diagram for this expression.

   (d) Solve the inequality:

$$\frac{x+3}{x-1}\leq 0$$

4. By drawing appropriate sign diagrams solve the following inequalities:

   (a) $\dfrac{x-2}{x+1}>0$

   (b) $\dfrac{x}{x+5}\leq 0$

   (c) $\dfrac{(x-2)(x+1)}{x+3}\geq 0$

   (d) $\dfrac{x(x-2)(x+4)}{(x+1)(x-3)}\leq 0$

   (e) $\dfrac{x^2-9}{x^2-5x}<0$

   (f) $\dfrac{x^2-x-12}{x^3+6x^2+8x}\geq 0$

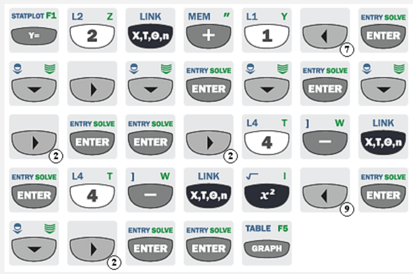
One can graph certain inequalities using the Ti-84. The Inequalz APP has more options for graphing inequalities. However this APP is not allowed in the IB examination.

EXAMPLE 1

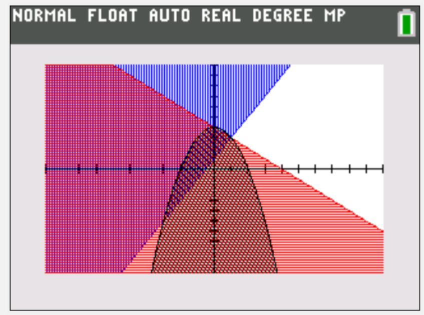
Draw the set bounded by the following inequalities:

$$\begin{cases} y > 2x+1 \\ y < 4-x \\ y < 4-x^2 \end{cases}$$

INPUT



OUTPUT



Note that the blue colour represents the inequality $y > 2x+1$, the red colour represents $y < 4-x$ and the black colour represents $y < 4-x^2$. So the set of points satisfying all three inequalities is the set coloured by all three colours.

SHORT TEST

1. [3 *points*]

Solve the following inequalities:

(a) $3 - 4x < 11$

(b) $15 - 7x \geq 2x^2$

2. [5 *points*]

Solve the following system of inequalities

$$\begin{cases} \frac{x+2}{2} - \frac{x-3}{3} > 1 \\ \\ x^2 \geq -5x \end{cases}$$

3. [4 *points*]

Consider the intervals $A =]-3, 2[$ and $B = [0, \infty[$. Find:

(a) $A \cap \mathbb{Z}$

(b) $A \cap B$

(c) $A - B$

(d) $B - A$

4. [3 *points*]

Shade the set of points satisfying the following system of inequalities:

$$\begin{cases} x \leq 1 \\ y > x \end{cases}$$

SHORT TEST
SOLUTIONS

1. [3 *points*]
Solve the following inequalities:

(a) $3 - 4x < 11$

$-4x < 8$

$x > -2$

(b) $15 - 7x \geq 2x^2$

$0 \geq 2x^2 + 7x - 15$

$0 \geq (2x-3)(x+5)$

$-5 \leq x \leq \frac{3}{2}$

2. [5 *points*]
Solve the following system of inequalities

$$\begin{cases} \frac{x+2}{2} - \frac{x-3}{3} > 1 \\ x^2 \geq -5x \end{cases}$$

$$\begin{cases} 3(x+2) - 2(x-3) > 6 \\ x^2 + 5x \geq 0 \end{cases}$$

$$\begin{cases} x + 12 > 6 \\ x(x+5) \geq 0 \end{cases}$$

$$\begin{cases} x > -6 \\ x \leq -5 \quad \text{or} \quad x \geq 0 \end{cases}$$

Which gives $-6 < x \leq -5$ or $x \geq 0$.

3. [4 *points*]
Consider the intervals $A =]-3, 2[$ and $B = [0, \infty[$. Find:

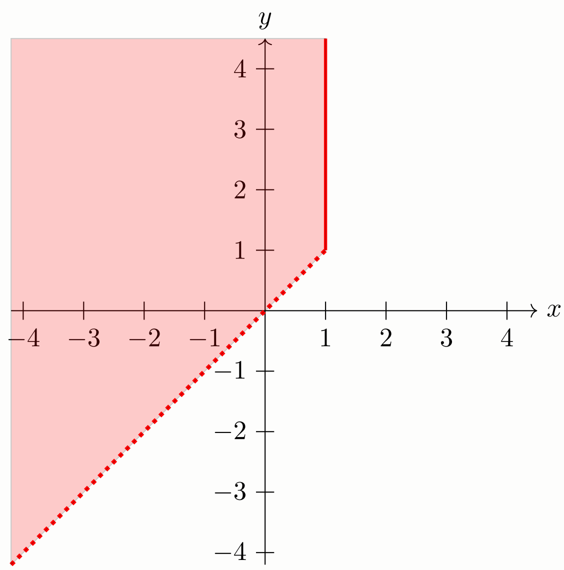
(a) $A \cap \mathbb{Z}$ (b) $A \cap B$ (c) $A - B$ (d) $B - A$

(a) $A \cap \mathbb{Z} = \{-2, -1, 0, 1\}$ (b) $A \cap B = [0, 2[$ (c) $A - B =]-3, 0[$ (d) $B - A = [2, \infty[$

4. [3 *points*]

Shade the set of points satisfying the following system of inequalities:

$$\begin{cases} x \leq 1 \\ y > x \end{cases}$$



1.6 Quadratic equations with parameters

NUMBER OF SOLUTIONS

Consider the following quadratic equations:

$$x^2+4x+3=0 \qquad x^2+4x+4=0 \qquad x^2+4x+5=0$$

The first equation has two solutions ($x=-1$ or $x=-3$), the second equation has one solution ($x=-2$, it is often referred to as **repeated** or **double** solution), the last equation has no real solutions. The number of real solutions of a quadratic equation depends on the sign of the discriminant:

$\Delta > 0$ two real solutions,

$\Delta = 0$ one (repeated) solution,

$\Delta < 0$ no real solutions.

Worked example 2.6.1

Find the number of real solutions of:

$$3x^2+5x+4=0$$

We have $a=3,\ b=5,\ c=4$, so:

$$\Delta = 5^2-4(3)(4) = -23 < 0$$

So the equation has no real solutions.

Exercise 2.6.1
Find the number of real solutions of the following equations (you do not need to find the solutions):

(a) $2x^2+5x+1=0$

(b) $\frac{1}{2}x^2-3x+2=0$

(c) $3x^2+5x+\sqrt{2}=0$

(d) $\frac{1}{2}x^2-4\sqrt{2}x+2\sqrt{3}=0$

(e) $\pi x^2+7x-\sqrt{7}=0$

(f) $\frac{1}{3}x^2+\frac{1}{2}x+\frac{1}{5}=0$

Worked example 2.6.2

For what values of k the following equation has two real solutions:

$$2x^2+6x+k=0$$

We have $a=2,\ b=6,\ c=k$, so:

$$\Delta = 6^2-4(2)(k) = 36-8k$$

We want $\Delta > 0$ in order to have two real solutions, so:

$$36-8k>0$$

which gives $8k<36$, so $k<4.5$. The equation has two real solutions provided that $k<4.5$.

Exercise 2.6.2a
For what values of p the following equation has exactly one real solution:

(a) $x^2 + 5x + p = 0$ (b) $3x^2 - x + 2p - 3 = 0$ (c) $\frac{1}{2}x^2 + x = p + 1$

(d) $x^2 + px + 3 - p = 0$ (e) $x^2 + (p + 3)x + 3p + 1 = 0$ (f) $x^2 + (p - 1)x + 2p + 3 = 0$

(g) $3x^2 + px + p + 7 = 2x$ (h) $x^2 + (p + 1)x + p = 2$ (i) $x^2 + (p - 4)x + p = 3$

Exercise 2.6.2b
For what values of q the following equation has exactly two real solutions:

(a) $x^2 - 2x + 3q = 0$ (b) $2x^2 - 3x + 2q - 1 = 0$ (c) $3x^2 - x = q$

(d) $x^2 + (q - 3)x + 6 - q = 0$ (e) $x^2 + (q - 1)x + q + 2 = 0$ (f) $x^2 + (5q + 1)x + 10 - q = 0$

(g) $x^2 + (q + 2)x + 3 = q$ (h) $x^2 + qx + q = 3x + 4$ (i) $x^2 + (q + 1)x + q = 2$

Exercise 2.6.2c
For what values of r the following equation has no real solutions:

(a) $2x^2 + 4x + 3r = 0$ (b) $\frac{1}{2}x^2 - 5x + 4r + 2 = 0$ (c) $2x^2 - 4x = r - 1$

(d) $x^2 + (2r + 4)x + 7r + 4 = 0$ (e) $x^2 + (r + 2)x + 3r - 2 = 0$ (f) $2x^2 + (r - 1)x + 7 - r = 0$

(g) $3x^2 + (r + 1)x + 2 = r$ (h) $x^2 + (3r + 1)x = 1 - r$ (i) $x^2 + rx + 2r = 5 - 2x$

VIETE'S FORMULAE

Consider a quadratic equation:

$$ax^2 + bx + c = 0$$

If the solutions to this equation are α and β, then the left hand side can be written in factored form as:

$$a(x - \alpha)(x - \beta) = 0$$

If we now expand this factored form and collect the like terms we get:

$$ax^2 - a(\alpha + \beta)x + a\alpha\beta = 0$$

But this should be the same equation as our original one:

$$ax^2 + bx + c = 0$$

So the coefficients should be the same. This gives us the following formulae:

$$-a(\alpha + \beta) = b \qquad \text{and} \qquad a\alpha\beta = c$$

Rearranging these we get, what is known as **Viete's formulae**:

$$\text{sum of solutions} = \alpha + \beta = -\frac{b}{a}$$

$$\text{product of solutions} = \alpha\beta = \frac{c}{a}$$

Exercise 2.6.3
Find the sum and product of solutions of the following equations:

(a) $x^2 + 5x + 1 = 0$

(b) $x^2 - 4x + 2 = 0$

(c) $x^2 - 4x = 0$

(d) $x^2 - 8 = 0$

(e) $3x^2 - 7x - 1 = 0$

(f) $\frac{1}{2}x^2 + 2x - 5 = 0$

(g) $3x^2 = x + 3$

(h) $4x^2 - 2x = 7$

(i) $\frac{1}{3}x^2 - x = 11$

Worked example 2.6.4

For what values of k the following equation has two real solutions, whose product is positive.

$$x^2 + (2 - 2k)x + 3k - 5 = 0$$

We have $a = 1,\ b = 2 - 2k,\ c = 3k - 5$, so:

$$\Delta = (2 - 2k)^2 - 4(3k - 5) = 4k^2 - 20k + 24$$

$$\Delta = 4(k - 2)(k - 3)$$

We want $\Delta > 0$, so $k < 2$ or $k > 3$.

We also want the product of solutions to be positive. We have:

$$\text{product} = \frac{c}{a} = 3k - 5$$

So we must have $3k - 5 > 0$, which gives $k > \frac{5}{3}$. Putting the two conditions together we get that:

$$\frac{5}{3} < k < 2 \qquad \text{or} \qquad k > 3$$

Exercise 2.6.4a
For what values of p the following equation has two real solutions, whose product is positive.

(a) $x^2 + 2px + 4p - 3 = 0$

(b) $x^2 + (1 - p)x + p = 1$

Exercise 2.6.4b
For what values of q the following equation has two real solutions, whose product is negative.

(a) $x^2 + (3q + 1)x + 8q + 1 = 0$

(b) $x^2 + 6qx + 2q = x$

Exercise 2.6.4c
For what values of r the following equation has two real solutions, whose sum is positive.

(a) $x^2 + (r - 4)x + r - 1 = 0$

(b) $x^2 + rx + 2r = 3$

Exercise 2.6.4d
For what values of s the following equation has two real solutions, whose sum is negative.

(a) $x^2 + sx + s + 8 = 0$

(b) $x^2 + 3sx + 5s = x - 1$

Exercise 2.6.4e
For what values of t the following equation has two real solutions, whose product is greater than their sum.

(a) $2x^2 + tx + 2t - 7 = 0$

(b) $x^2 + 3tx + t = 1$

Positive roots

In this investigation we will analyze the equation:

$$x^2 - kx + 3 - k = 0 \tag{1}$$

where k is a parameter, which can be any real number.

1. Solve equation (1) by hand for $k = -7$.

2. Use the GDC to solve the equation (1) for various other values of k. Can you find a value of k, for which the solution has two **positive** real solutions?

3. Show that equation (1) has two distinct real solutions only when $k < -6$ or $k > 2$.

   Let α and β be two real numbers.

4. If $\alpha + \beta > 0$, does it guarantee that both α and β are positive?

5. If $\alpha \times \beta > 0$, does it guarantee that both α and β are positive?

6. Explain why $\alpha + \beta > 0$ **and** $\alpha \times \beta > 0$ is a sufficient (and necessary) condition for both α and β to be positive.

7. Write down an expression in terms of k for the sum and product of roots of equation (1).

8. Find the set of all possible values of k for which equation (1) has two positive real roots.

9. Find the set of all possible values of k for which equation:

   $$x^2 + (3 - k)x + k = 0$$

   has two negative real roots.

SHORT TEST

1. [3 *points*]

State the number of solutions to the following equations:

(a) $2x^2 + 7x - 1 = 0$ (b) $3x^2 + 5 = 2x$ (c) $4x^2 = 4x - 1$

2. [6 *points*]

Find the possible values of k for which the following equations have (i) exactly one real solution, (ii) two real solutions, (iii) no real solutions:

(a) $x^2 + (k+3)x + 5k - 1 = 0$ (b) $x^2 + kx + 3k = 2$

3. [3 *points*]

Write down the sum and product of the following quadratic equations:

(a) $x^2 - 3x + 1 = 0$ (b) $2x^2 - 5 = 6x$ (c) $3x^2 = 2x + 1$

SHORT TEST

SOLUTIONS

1. [3 *points*]

State the number of solutions to the following equations:

(a) $2x^2 + 7x - 1 = 0$

$\Delta = 49 - 4(2)(-1) = 57 > 0$

2 real solutions

(b) $3x^2 + 5 = 2x$

$3x^2 - 2x + 5 = 0$

$\Delta = 4 - 4(3)(5) = -56 < 0$

no real solutions

(c) $4x^2 = 4x - 1$

$4x^2 - 4x + 1 = 0$

$\Delta = 16 - 4(4)(1) = 0$

one real solution

2. [6 *points*]

Find the possible values of k for which the following equations have (i) exactly one real solution, (ii) two real solutions, (iii) no real solutions:

(a) $x^2 + (k+3)x + 5k - 1 = 0$

$\Delta = (k+3)^2 - 4(5k-1)$

$\Delta = k^2 - 14k + 13$

$\Delta = (k-1)(k-13)$

one solution for $k = 1$ or $k = 13$

two real solutions for $k < 1$ or $k > 13$

no real solutions for $1 < k < 13$

(b) $x^2 + kx + 3k = 2$

$x^2 + kx + 3k - 2 = 0$

$\Delta = k^2 - 4(3k-2)$

$\Delta = k^2 - 12k + 8$

$\Delta = (k-6)^2 - 28$

one solution for $k = 6 - 2\sqrt{7}$ or $k = 6 + 2\sqrt{7}$

two real solutions for $k < 6 - 2\sqrt{7}$ or $k > 6 + 2\sqrt{7}$

no real solutions for $6 - 2\sqrt{7} < k < 6 + 2\sqrt{7}$

3. [3 *points*]

Write down the sum and product of the following quadratic equations:

(a) $x^2 - 3x + 1 = 0$

sum $= -\frac{-3}{1} = 3$

product $= \frac{1}{1} = 1$

(b) $2x^2 - 5 = 6x$

sum $= -\frac{-6}{2} = 3$

product $= \frac{-5}{2} = -\frac{5}{2}$

(c) $3x^2 = 2x + 1$

sum $= -\frac{-2}{3} = \frac{2}{3}$

product $= \frac{-1}{3} = -\frac{1}{3}$

1.7 Indices

Consider the following equation:

$$2^x = 8$$

Because the unknown x appears in the exponent only, this equation is an example of an **exponential equation**. We're looking for a number to which we need to raise 2 to get 8 as a result. This number is of course 3, so $x = 3$ is a solution to this equation. In fact it is the only solution.

Worked example 2.7.1

Solve the following equation:

$$8^x = 32$$

We start by writing both 8 and 32 as a power of the same number - in this case power of 2:

$$(2^3)^x = 2^5$$

Now, using the rules of indices, we get:

$$2^{3x} = 2^5$$

Equating exponents we get:

$$3x = 5$$

So $x = \frac{5}{3}$.

Exercise 2.7.1 Solve the following exponential equations:

(a) $4^x = 8$

(b) $16^x = 8$

(c) $64^x = \frac{1}{2}$

(d) $8^x = \frac{1}{4}$

(e) $4^x = \sqrt{2}$

(f) $32^x = \frac{1}{\sqrt{2}}$

(g) $9^x = 27$

(h) $81^x = 27$

(i) $27^x = \frac{1}{9}$

(j) $9^x = \sqrt{3}$

(k) $27^x = \sqrt[3]{9}$

(l) $81^x = \frac{1}{3\sqrt{3}}$

(m) $25^x = \frac{1}{5}$

(n) $\left(\frac{1}{125}\right)^x = \sqrt{5}$

(o) $125^x = \frac{1}{5\sqrt[3]{5}}$

In some cases the equation needs to be rearranged first in order to use the above method.

Worked example 2.7.2

Solve the following equation:

$$2 \times 9^x + 3 = 57$$

We start by subtracting 3 and then dividing by 2 both sides of the equation to get:

$$9^x = 27$$

Now, as before, we write both sides as a power of the same number (in this case 3):

$$(3^2)^x = 27^3$$

This gives:

$$3^{2x} = 27^3$$

So $2x = 3$ and this gives $x = \frac{3}{2}$.

Exercise 2.7.2 Solve the following equations:

(a) $3 \times 8^x - 1 = 11$

(b) $5 \times 27^x - 1 = 14$

(c) $4 \times 25^x + 20 = 520$

Worked example 2.7.3

Solve the following equation:

$$8^{x-3} = \left(\frac{1}{2}\right)^{x^2-1}$$

We write both sides as a power of the same number (in this case 2):

$$(2^3)^{x-3} = (2^{-1})^{x^2-1}$$

Using laws of indices we get:

$$2^{3x-9} = 2^{-x^2+1}$$

Which gives:

$$3x - 9 = -x^2 + 1$$

Moving all terms to one side we get a quadratic equation:

$$x^2 + 3x - 10 = 0$$

$$(x + 5)(x - 2) = 0$$

So $x = -5$ or $x = 2$.

Exercise 2.7.3 Solve the following equations:

(a) $4^{x+2} = 8^x$

(b) $27^{1-x} = 3^{2x+1}$

(c) $8^{x-2} = \left(\frac{1}{2}\right)^x$

(d) $25^{3-x} = \left(\frac{1}{5}\right)^{x+1}$

(e) $16^x = (\sqrt{2})^{x+1}$

(f) $64^{3-2x} = \left(\frac{1}{8}\right)^{x+1}$

(g) $49^{3x-1} = \left(\frac{1}{7}\right)^{x+2}$

(h) $125^{2x+1} = \left(\frac{1}{\sqrt{5}}\right)^{x+4}$

(i) $8^{x+3} = \left(\frac{1}{\sqrt[3]{2}}\right)^{x+3}$

(j) $81^{x+2} = \left(\frac{1}{27}\right)^{2x+1}$

(k) $128^{3-x} = \left(\frac{1}{2\sqrt{2}}\right)^{x+4}$

(l) $4^{x-2} = \left(\frac{1}{8\sqrt{2}}\right)^{x+3}$

(m) $8^{4-2x} = \left(\frac{1}{4}\right)^{3x-6}$

(n) $4^{x-4} = \left(\frac{1}{\sqrt{2}}\right)^{16-4x}$

(o) $125^{1-2x} = \left(\frac{1}{5}\right)^{6x+1}$

(p) $2^{x^2-1} = 16$

(q) $81^{x^2+1} = 243$

(r) $2^{x^2+3x} = \frac{1}{4}$

(s) $3^{x^2+2x} = 27$

(t) $625^{x-1} = 5^{x^2-1}$

(u) $9^{x+3} = \left(\frac{1}{3}\right)^{x^2+2}$

(v) $16^{x-2} = \left(\frac{1}{2}\right)^{x^2-4}$

(w) $\left(\frac{1}{5}\right)^{1-7x} = 25^{x^2+1}$

(x) $243 \times 3^x = \left(\frac{1}{\sqrt[3]{9}}\right)^{x-x^2}$

(y) $8^{4+x} = 4^x \times 2^{x^2}$

(z) $\left(\frac{1}{125}\right)^x = \sqrt{5}$

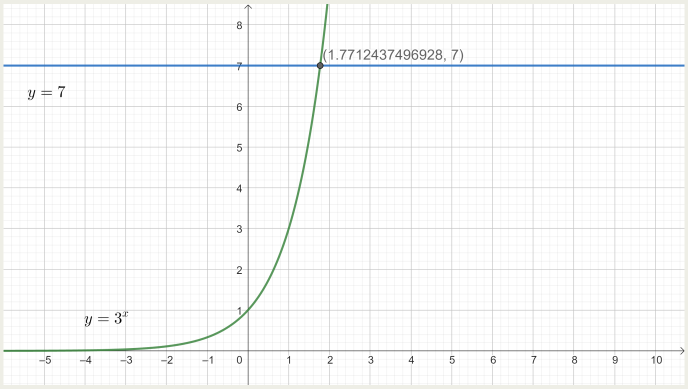
(zz) $\frac{16^x}{32} = 8 \times \left(\frac{1}{2}\right)^{2x-x^2}$

Worked example 2.7.4

Use a graph to solve:

$$3^x = 7$$

Sketching the graph of both $y = 3^x$ and $y = 7$ we get:



The graphs intersect at $(1.77124..., 7)$, so the solution is $x = 1.77$ (3 s.f.).

Exercise 2.7.4 Solve the following equations using graphs:

(a) $2^x = 6$

(b) $5^x = 3$

(c) $7^x = 10$

(d) $2^{x+1} = 45$

(e) $3^{2x-1} = 100$

(f) $5^{1-x} = 4$

(g) $4^x = \frac{1}{3}$

(h) $3^{x+1} = \frac{2}{5}$

(i) $5^{x-3} = \frac{1}{7}$

(j) $2 \times 5^x = 11$

(k) $3^x = 2^{x+1}$

(l) $4^x = \left(\frac{1}{5}\right)^{x-4}$

(m) $2 \times 3^x = 5^x - 2$

(n) $\left(\frac{1}{7}\right)^{x-1} = \frac{2^x}{5}$

(o) $10^x = 3^{x^2+1}$

Disguised quadratics

1. Solve the equation:

$$x^2 - 6x + 8 = 0$$

2. By letting $t = 2^x$ show that the equation

$$(2^x)^2 - 6 \times 2^x + 8 = 0$$

reduces to:

$$t^2 - 6t + 8 = 0$$

3. Show that $4^x = 2^{2x} = (2^x)^2$.

4. Hence solve the equation:

$$4^x - 6 \times 2^x + 8 = 0$$

5. Show that $2^{2x+1} = 2 \times (2^x)^2$ and hence, using an appropriate substitution solve:

$$2^{2x+1} - 17 \times 2^x + 8 = 0$$

6. By using an appropriate substitution solve:

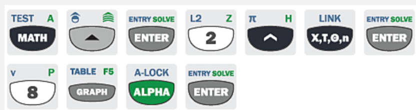
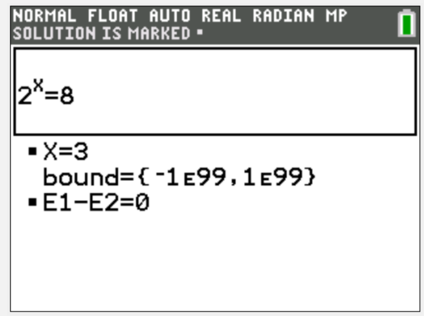
$$9^{x+1} - 10 \times 3^x + 1 = 0$$

7. Solve the equation:

$$4^{x+1} - 13 \times 2^x + 3 = 0$$

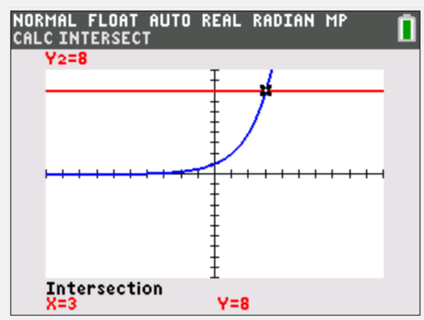
Exponential equations can be solved on GDC using the Numeric solver (as shown in example 1) or using graphs (as shown in example 2)

EXAMPLE 1

Solve:	INPUT	OUTPUT
$2^x = 8$		

Note that solver will only show you one answer at a time. You won't know, if there are more answers. Using the graphs allows you to check for more answers as long as you set up the display window appropriately.

EXAMPLE 2

Solve:	INPUT	OUTPUT
$2^x = 8$		

Note that the solution is the x coordinate of the point of intersection.

SHORT TEST

1. [9 *points*]

Solve the following equations without the use of technology:

(a) $9^{x+1} = \frac{1}{27}$

(b) $5 \times 4^{x-1} - 1 = \frac{3}{2}$

(c) $\dfrac{4}{3^{x+2}} = 36$

(d) $(\sqrt{2})^{2x^2+2} = \dfrac{16^x}{4}$

2. [4 *points*]

Solve the following by graphing appropriate functions on your GDC:

(a) $5^x = 30$

(b) $3^{x-1} = 2^{4-3x}$

SHORT TEST

SOLUTIONS

1. [9 *points*]

Solve the following equations without the use of technology:

(a) $9^{x+1} = \frac{1}{27}$

$3^{2x+2} = 3^{-3}$

$2x + 2 = -3$

$x = -\frac{5}{2}$

(b) $5 \times 4^{x-1} - 1 = \frac{3}{2}$

$5 \times 4^{x-1} = \frac{5}{2}$

$4^{x-1} = \frac{1}{2}$

$2^{2x-2} = 2^{-1}$

$2x - 2 = -1$

$x = \frac{1}{2}$

(c) $\dfrac{4}{3^{x+2}} = 36$

$\dfrac{1}{3^{x+2}} = 9$

$3^{-x-2} = 3^2$

$-x - 2 = 2$

$x = -4$

(d) $(\sqrt{2})^{2x^2+2} = \dfrac{16^x}{4}$

$2^{x^2+1} = \dfrac{2^{4x}}{2^2}$

$2^{x^2+1} = 2^{4x-2}$

$x^2 + 1 = 4x - 2$

$x^2 - 4x + 3 = 0$

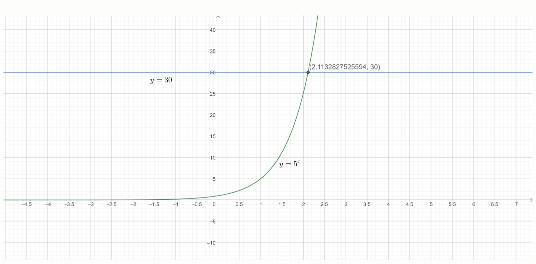
$(x-3)(x-1) = 0$

$x = 3 \quad \text{or} \quad x = 1$

2. [4 *points*]

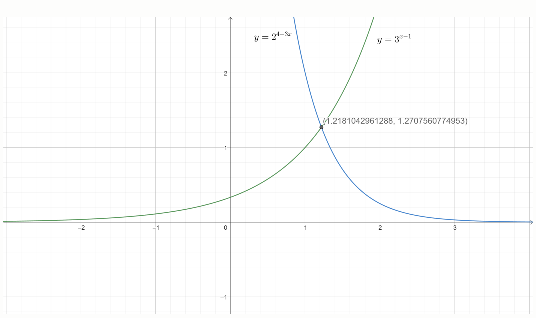
Solve the following by graphing appropriate functions on your GDC:

(a) $5^x = 30$



$x \approx 2.11$

(b) $3^{x-1} = 2^{4-3x}$



$x \approx 1.22$

1.8 Sequences

One can think of a sequence as a list of numbers. The order of the list is important (i.e. 1, 2, 3 and 3, 2, 1 are two different sequences). A number may appear on the list multiple times (1, 1 and 1, 1, 1 are again two different sequences). The number of elements on the list is called the length of a sequence (note that the list can be infinite in which case we have an infinite sequence). The elements on the list are often referred to as **terms**.

Consider the following sequence:

$$1,\ 5,\ 15,\ 51$$

This sequence has 4 terms. Its first term is 1, its second term is 5, its third term is 15 and its fourth term is 51. Lowercase letters are used to denote sequences and the lower indices are used to indicate the number of the term in the sequence. For the above sequence we have $a_1 = 1, a_2 = 5, a_3 = 15$ and $a_4 = 51$.

A sequence may be given by an explicit formula. For example the formula

$$a_n = 2^n - 1$$

gives the sequence:

$$1, \quad 3, \quad 7, \quad 15, \quad 31, ...$$

A sequence may also be given by an recursive formula. Consider the sequence defined by

$$\begin{cases} a_1 = 1 \\ a_{k+1} = 2a_k + 1 \qquad \text{for } k \geq 1 \end{cases}$$

The above definition gives us the first term and a rule to obtain any term from the previous one. Note that this definition also defines the sequence:

$$1, \quad 3, \quad 7, \quad 15, \quad 31, ...$$

You may realize an important advantage of an explicit formula over a recursive formula. The tenth (or any other) term of the sequence can be easily calculated using the explicit formula by substituting an appropriate value for n:

$$a_{10} = 2^{10} - 1 = 1023$$

However in order to calculate a_{10} using the recursive formula, we need to know a_9 and in order to calculate a_9, then a_8 is needed and so on.

Exercise 2.8.1 Write down the first five terms of the following sequences:

(a) $a_n = 3n + 2$

(b) $b_n = 3^{n-1}$

(c) $c_n = 2^n - n^2$

(d) $\begin{cases} d_1 = 5 \\ d_{k+1} = d_k - 2 \end{cases}$

(e) $\begin{cases} e_1 = 10 \\ e_{k+1} = 2e_k \end{cases}$

(f) $\begin{cases} f_1 = 1 \\ f_{k+1} = 3f_k - 1 \end{cases}$

(g) $\begin{cases} g_1 = 2 \\ g_2 = 1 \\ g_{k+2} = g_{k+1} - g_k \end{cases}$

(h) $\begin{cases} h_1 = 1 \\ h_2 = 2 \\ h_{k+2} = h_{k+1} \times h_k \end{cases}$

(i) $\begin{cases} i_1 = 2 \\ i_2 = 1 \\ i_{k+2} = i_{k+1}^{i_k} \end{cases}$

In many cases one can guess the explicit formula for a given sequence by simply looking at its initial terms. For example consider the sequence:

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, ...$$

It is fairly clear that the explicit formula for this sequence is $a_n = \frac{1}{n}$. Note that it is not as easy to figure out a recursive formula for the above sequence, but for example $\begin{cases} a_1 = 1 \\ a_{k+1} = \frac{a_k}{a_k+1} \end{cases}$ works.

Exercise 2.8.2 Guess an explicit formula and write down the next two terms for the following sequences:

(a) $2, \ 4, \ 8, \ 16, ...$

(b) $1, \ 3, \ 7, \ 15, ...$

(c) $8, \ 16, \ 32, \ 64, ...$

(d) $1, \ 4, \ 9, \ 16, ...$

(e) $2, \ 5, \ 10, \ 17, ...$

(f) $4, \ 9, \ 16, \ 25, ...$

(g) $5, \ 10, \ 15, \ 20, ...$

(h) $7, \ 12, \ 17, \ 22, ...$

(i) $15, \ 20, \ 25, \ 30, ...$

(j) $1, \ 8, \ 27, \ 64, ...$

(k) $3, \ 10, \ 29, \ 66, ...$

(l) $2, \ 10, \ 30, \ 68, ...$

LINEAR SEQUENCES

Consider the following sequences:

$$10,\ 16,\ 22,\ 28,\ 34, ...$$

$$25,\ 23,\ 21,\ 19,\ 17, ...$$

$$-21,\ -11,\ -1,\ 9,\ 19, ...$$

These are all examples of **arithmetic** (or **linear**) sequences. An arithmetic sequence is a sequence where the difference between consecutive terms is constant (we usually denote this constant difference with d). In the first example we have $d = 6$, in the second $d = -2$ and in the third $d = 10$. Note that (as in the second example) the difference may be negative, in which case the sequence is decreasing. Otherwise it is increasing (if $d > 0$) or constant (if $d = 0$).

Exercise 2.8.3 State the value of the common difference d and write down the next two terms of each of the following arithmetic sequences:

(a) $5,\ 8,\ 11,\ 14, ...$ (b) $13,\ 6,\ -1,\ -8, ...$ (c) $7,\ 7,\ 7,\ 7, ...$

(d) $\frac{1}{2},\ \frac{2}{3},\ \frac{5}{6},\ 1, ...$ (e) $3\sqrt{3},\ 5\sqrt{3},\ 7\sqrt{3},\ 9\sqrt{3}, ...$ (f) $2 + 3\sqrt{2},\ 3 + \sqrt{2},\ 4 - \sqrt{2},\ 5 - 3\sqrt{2}, ...$

Note that the general formula for an arithmetic sequence will be linear in n with gradient d i.e.:

$$u_n = d \times n + c$$

where c is a constant. The formulae for the three sequences listed at the top of the page are $a_n = 6n + 4$, $b_n = -2n + 27$ and $c_n = 10n - 29$ respectively.

Note that the name **arithmetic** comes from the fact that if a, b and c are three consecutive terms of an arithmetic sequence, then the midle term is an arithmetic mean of the other two terms, that is $b = \dfrac{a+c}{2}$.

Exercise 2.8.4 State the value of the common difference d and write down the first four terms of each of the following arithmetic sequences:

(a) $a_n = 5n - 1$ (b) $b_n = \frac{1}{2}n + \frac{3}{2}$ (c) $c_n = -3n + 12$

(d) $d_n = n\sqrt{2} + 4\sqrt{2}$ (e) $e_n = 5 - \frac{n}{3}$ (f) $f_n = 1 + \sqrt{3} - n(\sqrt{3} - 2)$

Exercise 2.8.5 The following sequences are arithmetic. Find the missing terms (denoted with x or y).

(a) $5,\ x,\ 17,\ 23, \ldots$

(b) $10,\ x,\ -4, \ldots$

(c) $x,\ 4,\ y,\ 20, \ldots$

(d) $x,\ y,\ 13,\ 9, \ldots$

(e) $11,\ x,\ y,\ 5, \ldots$

(f) $\frac{1}{3},\ x,\ y,\ \frac{7}{3}, \ldots$

Worked example 2.8.6

Find the general formula for the following arithmetic sequences:

(a) $7,\ 11,\ 15,\ 19, \ldots$

(a) We have $d = 4$, so the formula will be:

$$a_n = 4n + c$$

for some constant c. Now the first term is 7 ($a_1 = 7$), so substituting $n = 1$ should result in 7:

$$a_1 = 4 + c = 7$$

so $c = 3$ and the formula is $a_n = 4n + 3$.

(b) $10,\ 4,\ -2,\ -8, \ldots$

(b) We have $d = -6$, which gives the formula:

$$b_n = -6n + c$$

Now using the first term we get:

$$b_1 = -6 + c = 10$$

so $c = 16$ and the formula is $b_n = -6n + 16$.

Exercise 2.8.6 Find the general formula for the following arithmetic sequences:

(a) $2,\ 10,\ 18,\ 26, \ldots$

(b) $23,\ 25,\ 27,\ 29, \ldots$

(c) $37,\ 47,\ 57,\ 67, \ldots$

(d) $9,\ 7,\ 5,\ 3, \ldots$

(e) $15,\ 10,\ 5,\ 0, \ldots$

(f) $7,\ 11,\ 15,\ 19, \ldots$

(g) $\frac{2}{3},\ \frac{4}{3},\ 2,\ \frac{8}{3}, \ldots$

(h) $9,\ 8\frac{1}{2},\ 8,\ 7\frac{1}{2}, \ldots$

(i) $3\sqrt{2},\ 2\sqrt{2},\ \sqrt{2},\ 0, \ldots$

Worked example 2.8.7

Find the number of terms of the arithmetic sequence:

$$33,\ 29,\ 25,\ 21, ...,\ -67$$

We have $d = -4$, so the formula for this sequence is:

$$a_n = -4n + c$$

Using the fact that $a_1 = 33$ we to find c: $33 = -4 + c$. So $c = 37$ and the general formula for the sequence is:

$$a_n = -4n + 37$$

Now the last term a_n is equal to -67, so:

$$-4n + 37 = -67$$

Which gives $-4n = -104$, so $n = 26$, which means that the last term is the 26th term, so there are 26 terms.

Exercise 2.8.7 Find the number of terms of the following arithmetic sequence:

(a) $10,\ 16,\ 22,\ \ldots,\ 118$

(b) $5,\ 1,\ -3,\ \ldots,\ -79$

(c) $25,\ 22,\ 19,\ \ldots,\ -25$

(d) $\frac{1}{2},\ \frac{3}{4},\ 1,\ \ldots,\ 8\frac{1}{2}$

(e) $1\frac{2}{3},\ 2\frac{1}{3},\ 3,\ \ldots,\ 9\frac{2}{3}$

(f) $-\frac{2}{5},\ -\frac{1}{5},\ 0,\ \ldots,\ 2\frac{3}{5}$

(g) $3\sqrt{2},\ 5\sqrt{2},\ 7\sqrt{2},\ \ldots,\ 21\sqrt{2}$

(h) $\sqrt{3},\ 2+2\sqrt{3},\ 4+3\sqrt{3},\ \ldots,\ 26+14\sqrt{3}$

(i) $5-\sqrt{5},\ 3,\ 1+\sqrt{5},\ \ldots,\ 10\sqrt{5}-17$

(j) $a,\ 3a,\ 5a,\ \ldots,\ 21a$

(k) $a,\ 3a+b,\ 5a+2b,\ \ldots,\ 21a+10b$

(l) $x-2y,\ -x,\ 2y-3x,\ \ldots,\ 10y-11x$

Worked example 2.8.8

Find the first term of the arithmetic sequence:

$$111,\ 117,\ 123,\ 129, ...$$

which is greater than 1000.

We find the general formula for the n-th term of this sequence, which is:

$$a_n = 6n + 105$$

Now we solve:

$$6n + 105 > 1000$$

which gives $n > 149\frac{1}{6}$. So the first term greater than 1000 is $a_{150} = 1005$.

Exercise 2.8.8a Find the first term of the given arithmetic sequence which is greater than 500.

(a) $12,\ 21,\ 30,\ 39,\ldots$ (b) $-5,\ -1,\ 3,\ 7,\ldots$

Exercise 2.8.8b Find the first term of the given sequence which is smaller than 0.

(a) $120,\ 117,\ 114,\ 111,\ldots$ (b) $520,\ 507,\ 494,\ 481,\ldots$

The defining property of an arithmetic sequence is the constant difference between consecutive terms. This may be used to solve problems, where the terms of the sequence are expressed in terms of some unknown.

Worked example 2.8.9

The following are first three terms of an arithmetic sequence:

$$x+1, \quad 3x, \quad 4x+3$$

Find x and hence calculate these terms.

The difference between consecutive terms must be constant for an arithmetic sequence, so we must have:

$$3x-(x+1)=4x+3-3x$$

$$2x-1=x+3$$

Which gives $x=4$. The terms are:

$$5, \quad 12, \quad 19$$

Exercise 2.8.9a Find x, if the following terms are consecutive terms of an arithmetic sequence:

(a) $2x+1, \quad 4x, \quad 5x+4$ (b) $4x+4, \quad 6x, \quad 2x-1$ (c) $7x-1, \quad 3x, \quad x-3$

Exercise 2.8.9b Find x and y, if the following terms are consecutive terms of an arithmetic sequence:

(a) $2y,\ x,\ 16,\ 2x+y$ (b) $x+y,\ 2x-y,\ x+5y,\ 2x+3$ (c) $3x+1,\ 2x,\ x+y,\ 2y$

QUADRATIC AND CUBIC SEQUENCES

Consider the following sequence:

$$3, \ 7, \ 13, \ 21, \ 31, \ ...$$

This clearly is not an arithmetic sequence as the difference between consecutive terms is not constant ($7-3 \neq 13-7$). However if we write down the consecutive difference we get $7-3=4$, then $13-7=6$, then $21-13=8$ etc.:

$$4, \ 6, \ 8, \ 10, \ ...$$

We should notice that the above is an arithmetic sequence. That is the difference between consecutive differences of our original sequence is constant. Such sequences are called quadratic sequences and the general formula for them is:

$$a_n = an^2 + bn + c$$

where a, b and c are constants to be found. Let d be the constant difference between the difference (often referred to as **second difference**) of a quadratic sequence given by $a_n = an^2 + bn + c$. So we should have:

$$d = (a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) = a_{n+2} + a_n - 2a_{n+1}$$

This gives:

$$d = a(n+2)^2 + b(n+2) + c + an^2 + bn + c - 2(a(n+1)^2 + b(n+1) + c$$

This simplifies to:

$$d = an^2 + 4an + 4a + bn + 2b + c + an^2 + bn + c - 2(an^2 + 2an + a + bn + b + c)$$

Which finally gives:

$$d = 2a$$

So the coefficient a in the formula for a quadratic sequence is half of the second difference (difference between differences).

Note that if a_n is a quadratics sequence, then $a_n - an^2 = bn + c$ will be linear (arithmetic). We already know how to find a formula for an arithmetic sequence, so we should be able to find constants b and c.

Worked example 2.8.10

Find the general formula for the sequence:

$$3,\ 7,\ 13,\ 21,\ 31,\ ...$$

We already know that this will be a quadratic sequence as the difference between differences is constant. So the formula is of the form $a_n = an^2 + bn + c$. The second difference is equal to 2, so we must have $a = 1$.

Now we can proceed in two ways.

Method 1
Consider the sequence $a_n - n^2$ $(= bn + c)$:

$$2,\ 3,\ 4,\ 5,\ 6,\ ...$$

This is now an arithmetic sequence. We easily find the its formula is $n + 1$, so $b = 1$ and $c = 1$ and the formula for our original sequence is:

$$a_n = n^2 + n + 1$$

Method 2
We know that $a_1 = 3$ and $a_2 = 7$, we can use this information to set up a system of equations, by using the formula $a_n = n^2 + bn + c$ (we already know that $a = 1$):

$$\begin{cases} 1 + b + c = 3 \\ 4 + 2b + c = 7 \end{cases}$$

Solving this gives $b = 1$ and $c = 1$, so the formula is:

$$a_n = n^2 + n + 1$$

Alternative Method
Of course, if we forget that we can calculate a from the second difference, then we can simply set up 3 equations from $a_1 = 3$, $a_2 = 7$ and $a_3 = 13$:

$$\begin{cases} a + b + c = 3 \\ 4a + 2b + c = 7 \\ 9a + 3b + c = 13 \end{cases}$$

And solve for a, b and c to get the same result as above.

Exercise 2.8.10 Find the general formula for the following sequences:

(a) $2,\ 10,\ 26,\ 50,\ 82...$

(b) $1,\ 3,\ 7,\ 13,\ 21, ...$

(c) $3,\ 10,\ 21,\ 36,\ 55, ...$

(d) $10,\ 6,\ 0,\ -8,\ -18...$

(e) $2,\ 1,\ 2,\ 5,\ 10...$

(f) $5,\ 8,\ 7,\ 2,\ -9...$

(g) $\frac{1}{2},\ 2,\ \frac{11}{2},\ 11,\ \frac{37}{2}, ...$

(h) $8,\ 7\frac{1}{2},\ 6\frac{1}{2},\ 5,\ 3, ...$

(i) $5\sqrt{2},\ 4\sqrt{2},\ 2\sqrt{2},\ -\sqrt{2},\ -5\sqrt{2}, ...$

Consider now the following sequence:

$$3, \quad 9, \quad 27, \quad 63, \quad 123, \quad 213, \ldots$$

This time the third differences d_3 are constant. We start by writing the first differences:

$$6, \quad 18, \quad 36, \quad 60, \quad 90, \ldots$$

The second differences are:

$$12, \quad 18, \quad 24, \quad 30, \ldots$$

And finally the third differences:

$$6, \quad 6, \quad 6, \ldots$$

Sequences where the third difference is constant are given by the formula:

$$a_n = an^3 + bn^2 + cn + d$$

and are called cubic sequences.

Exercise 2.8.11 Show that for a cubic sequence in the form

$$a_n = an^3 + bn^2 + cn + d$$

we have $6a = d_3$ that is the coefficient a is one sixth of the third difference.

In order to find the other coefficients of the formula for a cubic sequence, we can proceed is similar fashion as with quadratic sequences. Note that if the sequence a_n is cubic, then the sequence $a_n - an^3$ will be quadratic, so once you found the coefficient a, you work with $a_n - an^3$.

Worked example 2.8.12

Find the general formula for the sequence:

$$3, \quad 9, \quad 27, \quad 63, \quad 123, \quad 213, \ldots$$

We already know that the third difference is 6, so $6a = 6$, which gives $a = 1$. Now consider the sequence $a_n - n^3$:

$$2, \quad 1, \quad 0, \quad -1, \quad -2, \quad -3, \ldots$$

Usually at this point we will get a quadratic sequence, but this one is linear, which means that $b = 0$, now c is the first difference so $c = -1$ and we must have $d = 3$, so finally:

$$a_n = n^3 - n + 3$$

Exercise 2.8.12 Find the general formula for the following sequences:

(a) $4,\ 16,\ 50,\ 118,\ 232,\ 404...$

(b) $2,\ 2,\ -6,\ -28,\ -70,\ -138...$

(c) $2,\ 17,\ 50,\ 107,\ 194,\ 317...$

(d) $6,\ 14,\ 12,\ -12,\ -70,\ -174...$

EXPONENTIAL SEQUENCES

Consider the following sequence:

$$3, \quad 6, \quad 12, \quad 24, \quad 48, \quad 96, ...$$

The differences of this sequence are never constant. However, one can quickly notice that the ratio of the consecutive terms is constant, that is:

$$\frac{6}{3} = \frac{12}{6} = \frac{24}{12} = \frac{48}{24} = \frac{96}{48}$$

A sequence for which the ratio of consecutive terms is constant is called an **exponential** (or **geometric**) sequence.

Let a_1 is the first term and r the constant ratio of a gemeotric sequence (in the example above, we have $a_1 = 3$ and $r = 2$). The terms of the sequence are:

$$a_1, \quad a_1 r, \quad a_1 r^2, \quad a_1 r^3, \quad a_1 r^4, ...$$

That is we have that the second term $a_2 = a_1 r$, the third term $a_3 = a_1 r^2$, etc. and the general formula is thus:

$$a_n = a_1 r^{n-1}$$

For the example above we get the formula

$$a_n = 3 \times 2^{n-1}$$

Exercise 2.8.13 Write down the ratio, r, and the general formula for the following geometric sequences:

(a) $7, \ 21, \ 63, \ 189, ...$

(b) $8, \ 4, \ 2, \ 1, ...$

(c) $90, \ 30, \ 10, \ 3\frac{1}{3}, ...$

(d) $3, \ 30, \ 300, \ 3000, ...$

(e) $5, \ -5, \ 5, \ -5, ...$

(f) $7, \ -14, \ 28, \ -56, ...$

(g) $12, \ -6, \ 3, \ -1.5, ...$

(h) $-1, \ -5, \ -25, \ -125, ...$

(i) $2, \ 2\sqrt{2}, \ 4, \ 4\sqrt{2}, ...$

(j) $q, \ 3q^2, \ 9q^3, \ 27q^4, ...$

(k) $x, \ x^2y, \ x^3y^2, \ x^4y^3, ...$

(l) $\frac{a}{b}, \ 1, \ \frac{b}{a}, \ \frac{b^2}{a^2}, ...$

Worked example 2.8.14

Find the first term of the following geometric sequence that is greater than 25000:

$$16,\ 40,\ 100,\ 250,\ 625, ...$$

The ratio of the sequence is 2.5, so the general formula for the sequence is:

$$u_n = 16 \times (2.5)^{n-1}$$

Now we want:

$$16 \times (2.5)^{n-1} > 25000$$

We can solve this inequality on the GDC using graphs (graphing both $y = 16 \times (2.5)^{n-1}$ and $y = 25000$) or tables. We get that $n > 9.02588...$, so the first term greater than 25000 is $u_{10} \approx 61035$ (the previous term $u_9 \approx 24414$).

Exercise 2.8.14 Find the first term of the following geometric sequences that is greater than 40 000.

(a) $3,\ 12,\ 48,\ 192, ...$

(b) $100,\ 110,\ 121,\ 133.1, ...$

(c) $2,\ -3,\ 4.5,\ -6.75, ...$

(d) $3,\ 3\sqrt{2},\ 6,\ 6\sqrt{2}, ...$

(e) $\frac{1}{8},\ \frac{1}{4},\ \frac{1}{2},\ 1, ...$

(f) $16,\ 20,\ 25,\ 31.25, ...$

(g) $2,\ \pi,\ \frac{\pi^2}{2},\ \frac{\pi^3}{4}, ...$

(h) $2,\ \frac{2\sqrt{3}}{\sqrt{2}},\ 3,\ \frac{3\sqrt{3}}{\sqrt{2}}, ...$

(i) $\sqrt[3]{3},\ \sqrt[3]{9},\ 3,\ 3\sqrt[3]{3}, ...$

Worked example 2.8.15

Find the value of x for which the following terms are consecutive terms of a geometric sequence:

$$x+3, \quad 2x+2, \quad 4x-2$$

In a geometric sequence the ratio must be constant, so we must have:

$$\frac{2x+2}{x+3} = \frac{4x-2}{2x+2}$$

By multiplying both sides by $x+3$ and $2x+2$ we get:

$$4x^2 + 8x + 4 = 4x^2 + 10x - 6$$

This gives $10 = 2x$, so $x = 5$ and the sequence is $8, \quad 12, \quad 18$.

Exercise 2.8.15 Find the value of x for which the following terms are consecutive terms of a geometric sequence:

(a) $4x, \quad x, \quad \dfrac{x}{2}+1$

(b) $x-3, \quad 12, \quad 5x+1$

(c) $2x, \quad x-1, \quad \dfrac{2}{3}$

(d) $x+1, \quad x-1, \quad 3-x$

(e) $x-4, \quad 8-x, \quad 2x-1$

(f) $2, \quad x+3, \quad x^2+1$

Worked example 2.8.16

Find the values of x and y given that

$$x, \quad 4, \quad y$$

are consecutive terms of a geometric sequence and

$$x, \quad 4, \quad y-2$$

are consecutive terms of an arithmetic sequence.

Using the fact that $x, \quad 4, \quad y$ is geometric and $x, \quad 4, \quad y-2$ is arithmetic we can form a system of equations:

$$\begin{cases} \dfrac{4}{x} = \dfrac{y}{4} \\ 4-x = y-2-4 \end{cases}$$

$$\begin{cases} xy = 16 \\ 10-x = y \end{cases}$$

Substituting the second equation into the first one and rearranging results in the equation $x^2 - 10x + 16 = 0$, which gives $x = 2$ or $x = 8$. If $x = 2$, then $y = 8$ and if $x = 8$, then $y = 2$.

Exercise 2.8.17 For the following sequences decide if they are linear (arithmetic), quadratic, cubic, exponential (geometric) or neither.

(a) $10,\ 7,\ 4,\ 1,\ -2...$

(b) $5,\ 10,\ 20,\ 40,\ 80, ...$

(c) $6,\ 8,\ 11,\ 15,\ 20, ...$

(d) $-1,\ 11,\ 39,\ 89,\ 167...$

(e) $24,\ 12,\ 6,\ 3,\ 1.5...$

(f) $100,\ 80,\ 55,\ 25,\ -10...$

(g) $2,\ 3,\ 5,\ 7,\ 11, ...$

(h) $2,\ 4,\ 8,\ 16,\ 31, ...$

(i) $3,\ 3\sqrt{2},\ 6,\ 6\sqrt{2},\ 12...$

Fibonacci sequence

The Fibonacci sequence can be defined by the recursive rule:

$$F_n = \begin{cases} 1 & \text{for } n = 1 \text{ or } n = 2 \\ F_{n-1} + F_{n-2} & \text{for } n \geq 3 \end{cases}$$

The goal of this investigation is to find an explicit formula for F_n.

1. The first seven terms of the Fibonacci sequence are given below:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13, \ldots$$

   Write the next three terms of this sequence.

2. Let ϕ and ψ be the solutions (with $\phi > \psi$) to the equation:

$$x^2 = x + 1 \tag{1}$$

   Calculate the exact value of ϕ and ψ.

3. Using equation (1) show that

$$\phi^3 = 2\phi + 1 \qquad \text{and} \qquad \phi^4 = 3\phi + 2$$

4. Write similar expressions for ϕ^5 and ψ^6.

5. Generalize your answers to parts 3 and 4 to write down similar expressions for ϕ^n and ψ^n.

6. Find an expression for F_n using the system of equations:

$$\begin{cases} \phi^n = F_n\phi + F_{n-1} \\ \psi^n = F_n\psi + F_{n-1} \end{cases}$$

7. Check your formula for the initial terms of the Fibonacci sequence given in 1.

   The formula

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

   is known as the Binet's formula.

SHORT TEST

1. [6 *points*]

Find the general formula for each of the following sequences:

(a) $15, \quad 11, \quad 7, \quad 3, \quad -1, \quad -5, ...$

(b) $-3, \quad -3, \quad 1, \quad 9, \quad 21, \quad 37, ...$

(c) $-4, \quad -5, \quad 2, \quad 23, \quad 64, \quad 131, ...$

(d) $40, \quad -20, \quad 10, \quad -5, \quad 2.5, \quad -1.25, ...$

2. [4 *points*]

Find the first term of the given sequence which is greater than 20 000:

(a) $2, \quad 6, \quad 18, \quad 54, ...$

(b) $8, \quad 12, \quad 18, \quad 27, ...$

3. [4 *points*]

Find the value(s) of x given that the following terms are consecutive terms of a sequence which is arithmetic (part (a)), geometric (part (b)):

(a) $2x-3, \quad x-1, \quad 6-x$

(b) $2x+2, \quad x-2, \quad \frac{x}{4}$

SHORT TEST

SOLUTIONS

1. [6 *points*]

Find the general formula for each of the following sequences:

(a) $15, \quad 11, \quad 7, \quad 3, \quad -1, \quad -5, ...$

The difference is constant and equal to -4, so the sequence will be of the form $u_n = -4n + c$. Using the fact that $u_1 = 15$, we get that $c = 19$, so the general formula is $u_n = -4n + 19$.

(b) $-3, \quad -3, \quad 1, \quad 9, \quad 21, \quad 37, ...$

The differences are:

$$0, \quad 4, \quad 8, \quad 12, \quad 16, ..$$

The second differences are constant and equal to 4. This means that we have a quadratic sequence $an^2 + bn + c$ with $a = \frac{4}{2} = 2$. So $v_n = 2n^2 + bn + c$. Now we consider $v_n - 2n^2$ and we get:

$$-5, \quad -11, \quad -17, \quad -23, \quad -29, \quad -35, ...$$

We get a linear sequence with difference equal to -6, given that the first term is -5 it must be of the form $-6n + 1$, so the general term for the original sequence is $v_n = 2n^2 - 6n + 1$.

(c)$-4, \quad 0, \quad 14, \quad 44, \quad 96, \quad 176, ...$

This time the third differences are constant, so this is a cubic seqence. The third differences are 6, so the coefficient of n^3 is 1. Considering the sequence $w_n - n^3$ we get a quadratic sequence $-n^2 - 4$, so finally $w_n = n^3 - n^2 - 4$.

(d) $40, \quad -20, \quad 10, \quad -5, \quad 2.5, \quad -1.25, ...$

The ratio is constant (and equal to $-\frac{1}{2}$), so we have a geometric sequence, the first term is 40, so the general formula is $x_n = 40 \times (-\frac{1}{2})^{n-1}$.

2. [4 *points*]

Find the first term of the given sequence which is greater than 20 000:

(a) $2, \quad 6, \quad 18, \quad 54, ...$

(b) $8, \quad 12, \quad 18, \quad 27, ...$

(a) The ratio of the sequence is constant (and equal to 3), so this is a geometric sequence. Since the first term is 2, the formula for the sequence is:

$$u_n = 2 \times 3^{n-1}$$

We want $2 \times 3^{n-1} > 20000$. Using GDC we get that $n > 9.38...$, so $u_{10} = 39366$ is the first term of the sequence greater than 20 000 ($u_9 = 13122$).

(b) The ratio is again constant (and equal to 1.5), so this is also a geometric sequence. The formula for this sequence is:

$$v_n = 8 \times (1.5)^{n-1}$$

We want $8 \times (1.5)^{n-1} > 20000$. Using GDC we get $n > 20.296...$, so $v_{21} \approx 26602$ is the first term of the sequence greater than 20000 ($v_{20} \approx 17735$).

3. [4 *points*]

Find the value(s) of x given that the following terms are consecutive terms of a sequence which is arithmetic (part (a)), geometric (part (b)):

(a) $2x-3, \quad x-1, \quad 6-x$

(b) $2x+2, \quad x-2, \quad \dfrac{x}{4}$

(a) The sequence is arithmetic, so the difference between consecutive terms must be constant:

$$(x-1)-(2x-3)=(6-x)-(x-1)$$

Solving the above equation gives $x=5$ and the terms of the sequence are $7,\ 4,\ 1$.

(b) The sequence is geometric, so the ratio between consecutive terms must be constant:

$$\frac{x-2}{2x+2}=\frac{\frac{x}{4}}{x-2}$$

After cross multiplying and moving all terms to one side we get:

$$x^2-9x+8=0$$

Which factorizes into $(x-1)(x-8)=0$, so $x=1$ or $x=8$ and the possible sequences are: $4,\ -1,\ \frac{1}{4}$ or $18,\ 6,\ 2$.

1.9 Absolute value

$|x|$ denotes the absolute value of a number. If x is non-negative, then $|x| = x$, so for example $|3| = 3$, $|\pi| = \pi$ and $|2 - \sqrt{2}| = 2 - \sqrt{2}$. If x is negative, then $|x| = -x$, so $|-3| = 3$ and $|1 - \sqrt{2}| = \sqrt{2} - 1$. Another way to think about the absolute value is that $|x|$ denotes the distance on the number line from 0 to x, so $|5| = 5 = |-5|$, because both 5 and -5 are 5 units away from 0.

Worked example 2.9.1

Simplify the following expression:

$|2\sqrt{2}-3|-|3\sqrt{2}-4|+2|5\sqrt{2}-7|$

We have:

$$2\sqrt{2} = \sqrt{8} < \sqrt{9} = 3$$

So $2\sqrt{2}-3$ is negative, which means that $|2\sqrt{2}-3| = -2\sqrt{2}+3$

Similarly:

$$3\sqrt{2} = \sqrt{18} > \sqrt{16} = 4$$

So $|3\sqrt{2} - 4| = 3\sqrt{2} - 4$

And finally:

$$5\sqrt{2} = \sqrt{50} > \sqrt{49} = 7$$

So $|5\sqrt{2} - 7| = 5\sqrt{2} - 7$.

We then have:

$|2\sqrt{2} - 3| - |3\sqrt{2} - 4| + 2|5\sqrt{2} - 7| =$

$= -2\sqrt{2} + 3 - (3\sqrt{2} - 4) + 2(5\sqrt{2} - 7) =$

$= -2\sqrt{2} + 3 - 3\sqrt{2} + 4 + 10\sqrt{2} - 14 =$

$= 5\sqrt{2} - 7 =$

Exercise 2.9.1 Simplify the following expressions:

(a) $2|\sqrt{3} - 2| + |2\sqrt{3} - 3| - 2|3\sqrt{3} - 5|$

(b) $|\sqrt{5} - 2| - 3|2\sqrt{5} - 4| + 2|3\sqrt{5} - 9|$

(c) $2|4\sqrt{7} - 10| - |3\sqrt{7} - 9| + |6\sqrt{7} - 15|$

(d) $|2\sqrt{11} - 6| - |3\sqrt{11} - 10| + 2|4\sqrt{11} - 13|$

(e) $2|\sqrt{3} - 2| + |2\sqrt{3} - 3| - 2|3\sqrt{3} - 5|$

(f) $|7\sqrt{2} - 9| + 2|8\sqrt{2} - 12| - |10\sqrt{2} - 14|$

Consider the expression $\sqrt{x^2}$. If x is nonnegative, then $\sqrt{x^2} = x$, but if x is negative, then we get $\sqrt{x^2} = -x$. For example $\sqrt{(-3)^2} = \sqrt{9} = 3$. Therefore $\sqrt{x^2} = |x|$.

Worked example 2.9.2

Evaluate:

(a) $\sqrt{(7-3\sqrt{5})^2}$

(b) $\sqrt{20-6\sqrt{11}}$

(a) We have:

$$\sqrt{(7-3\sqrt{5})^2} = |7-3\sqrt{5}|$$

and since $3\sqrt{5} = \sqrt{45} < \sqrt{49} = 7$ we get that $7-3\sqrt{5}$ is positive and hence:

$$\sqrt{(7-3\sqrt{5})^2} = 7-3\sqrt{5}$$

(b) We would like to write $20-6\sqrt{11}$ as something squared. By comparing $20-6\sqrt{11}$ to $a^2-2ab+b^2$, we notice that if we take $a=3$ and $b=\sqrt{11}$, then $a^2+b^2=20$ and $2ab=6\sqrt{11}$, so we can write:

$$20-6\sqrt{11} = (3-\sqrt{11})^2$$

This means that:

$$\sqrt{20-6\sqrt{11}} = \sqrt{(3-\sqrt{11})^2} = |3-\sqrt{11}|$$

Since $\sqrt{11} > \sqrt{9} = 3$, so $3-\sqrt{11}$ is negative and thus:

$$\sqrt{20-6\sqrt{11}} = -3+\sqrt{11}$$

Exercise 2.9.2 Evaluate:

(a) $\sqrt{(11-5\sqrt{3})^2}$

(b) $\sqrt{(7\sqrt{2}-10)^2}$

(c) $\sqrt{(7-3\sqrt{6})^2}$

(d) $\sqrt{6-2\sqrt{5}}$

(e) $\sqrt{21+8\sqrt{5}}$

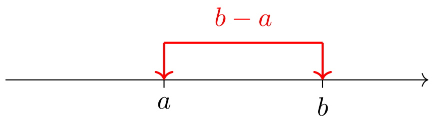
(f) $\sqrt{7-4\sqrt{3}}$

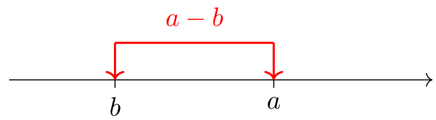
(g) $\sqrt{33-20\sqrt{2}}$

(h) $\sqrt{52-14\sqrt{3}}$

(i) $\sqrt{11-6\sqrt{2}}$

Note that the distance between two numbers a and b on the number line is $a - b$ if $a \geq b$ or $b - a$ if $b > a$, so it can be written as $|a - b|$. We clearly have $|a - b| = |b - a|$, that is the distance between a and b is the same as the distance between b and a.



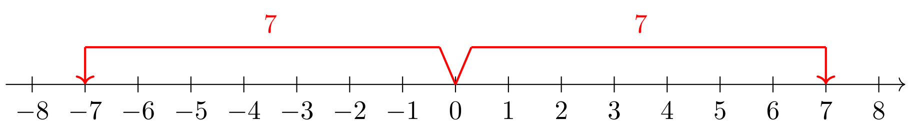


This means that an expression $|x - 5|$ can be thought of as a distance between numbers x and 5 on the number line. Similalrly $|x+2| = |x-(-2)|$ is the distance between x and -2, and $|x| = |x-0|$ is the distance between x and 0.

Consider the equation:

$$|x| = 7$$

We're looking for a number whose absolute value is 7. There are two such numbers 7 and -7 and these are the solutions to our equation. Alternatively we can think of a number whose distance from an 0 on the number line is 7:



Now consider the following equation:

$$|x - 3| = 4$$

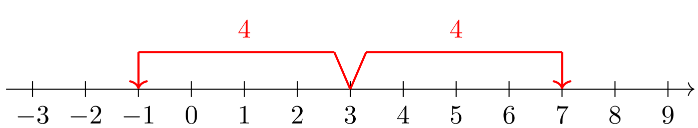
We can consider the expression inside the absolute value - it has to be either 4 or -4, so we must have:

$$x - 3 = 4 \qquad \text{or} \qquad x - 3 = -4$$

So:

$$x = 7 \qquad \text{or} \qquad x = -1$$

We can also think of the equation $|x-3| = 4$ as indicating that the distance between x and 3 on the number line has to be 4:



Worked example 2.9.3

Solve the equation:

$$3|x+1|-1=8$$

We start by adding 1 to both sides and then dividing both sides by 3 to get:

$$|x+1|=3$$

Now we can proceed in two ways:

Method 1

The expression inside the absolute value has to be 3 or -3, so we get:

$$x+1=3 \qquad \text{or} \qquad x+1=-3$$

So:

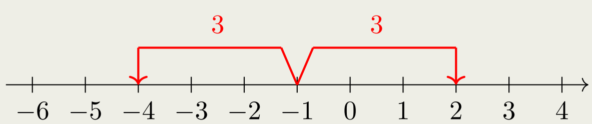
$$x=2 \qquad \text{or} \qquad x=-4$$

Method 2

We rewrite the equation as:

$$|x-(-1)|=3$$

So the distance between x and -1 is 3. On the number line this gives:



Exercise 2.9.3 Solve the following equations:

(a) $4|x|-5=11$

(b) $5|x-1|-2=8$

(c) $\frac{1}{2}|x+3|+1=2$

(d) $\dfrac{|x-1|+3}{2}=4$

(e) $3|x+2|+7=1$

(f) $|x+3|+3|x+3|=2$

(g) $|x-2|+2|2-x|=9$

(h) $\dfrac{|x-1|}{2}-\dfrac{|1-x|}{3}=1$

(i) $\dfrac{|2x-10|}{3}-\dfrac{|5-x|}{3}=2$

(j) $\dfrac{|3x-6|}{4}+\dfrac{|2-x|}{2}=2$

(k) $\dfrac{|2x-8|}{3}+\dfrac{|12-3x|}{4}=2$

(l) $\dfrac{|x-1|}{5}-|1-x|=2+|2x-2|$

(m) $\sqrt{(x+1)^2}+2|x+1|=9$

(n) $\sqrt{(2x-4)^2}-|2-x|=3$

(o) $\frac{|x-3|}{2}+\sqrt{x^2-6x+9}=3$

Consider the inequality:

$$|x-2| \leq 4$$

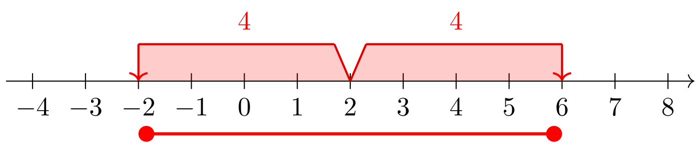
We can consider the expression inside the absolute value and note that it must be smaller or equal to 4 **and** also greater or equal than -4, so we must have:

$$-4 \leq x-2 \leq 4$$

Adding 2 to all sides of the inequalities we get:

$$-2 \leq x \leq 6$$

We can also note that $|x-2|$ is the distance between x and 2 on the number line and this distance cannot exceed 4, so we must have:



So the inequality is satisfied by all numbers between -2 and 6 (inclusive). Using interval notation we must have $x \in [-2, 6]$.

Now consider the inequality:

$$|x+1| > 2$$

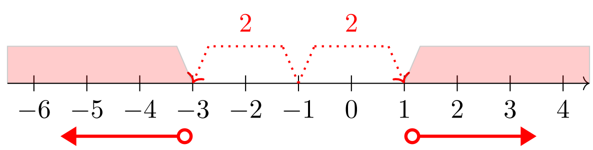
The expression inside the absolute value must be greater than 2 **or** smaller than -2, so:

$$x+1 > 2 \qquad \text{or} \qquad x+1 < -2$$

Which gives:

$$x > 1 \qquad \text{or} \qquad x < -3$$

Alternatively we can note that the distance between x and -1 (remember $|x+1| = |x-(-1)|$) has to be greater than 2:



Exercise 2.9.4 Solve the following inequalities:

(a) $|x-5| > 1$

(b) $|x+3| \leq 0.5$

(c) $|x-4| \geq 3$

(d) $\sqrt{(x+2)^2} \leq 3$

(e) $\sqrt{x^2-6x+9} < 1$

(f) $\sqrt{x^2+2x+1} \leq 4$

(g) $|3-x| + 2|x-3| \geq 6$

(h) $|3-3x| - \sqrt{x^2-2x+1} \leq 8$

(i) $\dfrac{|x-2|}{3} - \dfrac{|2-x|}{4} < 1$

(j) $|x-7|+4>2$

(k) $|x-3|+2\leq 1$

(l) $\dfrac{|x-4|+2}{2}\geq 1$

(m) $\dfrac{|2x-2|}{3}-|1-x|\leq 1$

(n) $|x-3|+2<\dfrac{|3-x|}{2}$

(o) $|x+2|+|2x+4|\leq|4x+8|+1$

(p) $\dfrac{|2-x|}{3}-1\geq|2x-4|$

(q) $|1+\frac{x}{2}|\leq 8+|x+2|$

(r) $\sqrt{x^2-10x+25}>\dfrac{|5-x|-1}{2}$

(s) $|2x-5|+|4x-10|<6$

(t) $\dfrac{|x-1|}{3}>\dfrac{|1-x|+1}{4}$

(u) $|4-x|+\dfrac{\sqrt{x^2-8x+16}}{3}\leq 3$

(v) $\sqrt{x^2+2x\sqrt{2}+2}\leq 3\sqrt{2}$

(w) $\sqrt{x^2-2x\sqrt{3}+3}<\sqrt{3}$

(x) $\dfrac{|x-1|}{2}\leq 3-\sqrt{x^2-2x+1}$

SHORT TEST

1. [4 *points*]

Simplify the following expressions:

(a) $|3-2\sqrt{2}|-2|4-3\sqrt{2}|+3|7-5\sqrt{2}|$

(b) $\sqrt{19-8\sqrt{3}}+2|5-3\sqrt{3}|$

2. [4 *points*]

Solve the following equations:

(a) $5|x+1|+1=11$

(b) $\dfrac{|x-1|}{2}-\dfrac{|2-2x|}{3}=|3-3x|-1$

3. [7 *points*]

Solve the following inequalities:

(a) $4|x-3|+7>15$

(b) $3-2|x+2|\leq 9$

(c) $\sqrt{x^2-4x+4}+\dfrac{|2-x|}{2}>|2x-4|-1$

SHORT TEST

SOLUTIONS

1. [4 *points*]

Simplify the following expressions:

(a) $|3-2\sqrt{2}|-2|4-3\sqrt{2}|+3|7-5\sqrt{2}|$

$= 3-2\sqrt{2}-2(-4+3\sqrt{2})+3(-7+5\sqrt{2}) =$

$= 3-2\sqrt{2}+8-6\sqrt{2}-21+15\sqrt{2} =$

$= 7\sqrt{2}-10$

(b) $\sqrt{19-8\sqrt{3}}+2|5-3\sqrt{3}|$

$= \sqrt{(4-\sqrt{3})^2}+2(-5+3\sqrt{3}) =$

$= 4-\sqrt{3}-10+6\sqrt{3} =$

$= 5\sqrt{3}-6$

2. [4 *points*]

Solve the following equations:

(a) $5|x+1|+1=11$

$|x+1|=2$

$x+1=2 \quad \text{or} \quad x+1=-2$

$x=1 \quad \text{or} \quad x=-3$

(b) $\dfrac{|x-1|}{2}-\dfrac{|2-2x|}{3}=|3-3x|-1$

$3|x-1|-2|2-2x|=6|3-3x|-6$

$3|x-1|-4|x-1|=18|x-1|-6$

$\frac{6}{19}=|x-1|$

$x-1=\frac{6}{19} \quad \text{or} \quad x-1=-\frac{6}{19}$

$x=\frac{25}{19} \quad \text{or} \quad x=\frac{13}{19}$

3. [7 *points*]

Solve the following inequalities:

(a) $4|x-3|+7>15$

$|x-3|>2$

$x-3<-2 \quad \text{or} \quad x-3>2$

$x<1$ or $x>5$

(b) $3-2|x+2|\le 9$

(b) $3-2|x+2|\le 9$

$|x+2|\ge -3$

$x\in\mathbb{R}$

(c) $\sqrt{x^2-4x+4}+\dfrac{|2-x|}{2}>|2x-4|-1$

$|x-2|+\frac{|x-2|}{2}>2|x-2|-1$

$1>0.5|x-2|$

$2>|x-2|$

$-2<x-2<2$

$0<x<4$

1.10 End of unit test

TEST 2

- The test consists of two sections. In section A calculators are **not allowed**. GDC is required for section B.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- The maximum mark for this test is [**36 + 36 marks**].
- Time allowed is **90 minutes**.
- Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to **show all working**.

SECTION A

1. [10 *points*]

Solve the following equations or systems of equations:

(a) $3x - 2\sqrt{2} = x\sqrt{2} + 1$ [2]

(b) $x^2 + 5x = 14$ [2]

(c) $4^{x-1} = 8^{x+1}$ [2]

(d) $\begin{cases} 4x - y = 3 \\ 3x + 2y = 16 \end{cases}$ [2]

(e) $2|x-3| + 4|3-x| - 7 = 5$ [2]

---

2. [10 *points*]

Solve the following inequalities and systems of inequalities:

(a) $\dfrac{x-1}{3} - \dfrac{2x+1}{4} \leq \dfrac{1}{2}$ [2]

(b) $2\sqrt{x^2+4x+4}-1 \geq 9$ [2]

(c) $x^2+6x+13 \geq 0$ [2]

(d) $\begin{cases} x^2 \geq 4x \\ x^2 - x < 6 \end{cases}$ [4]

3. [5 *points*]

The first three terms of an arithmetic sequence are:

$$x+2, \quad 3x, \quad 4x+3$$

(a) Find the value of x. [2]

(b) Write down the fourth term of this sequence. [1]

(c) Find the general formula for the n-th term of this sequence. [2]

---

4. [4 *points*]

Simplify the following algebraic fraction. State all necessary assumptions.

$$\frac{x^3+3x^2-4x-12}{x^4-13x^2+39}$$

5. [7 *points*]

Consider the equation:

$$x^2 + (m-3)x + 6 = m$$

(a) Find all possible values of parameter m for which the equation has two distinct real solutions. [3]

(b) Write down the product and the sum of the solutions in terms of m. [2]

(c) Find the solutions for $m = 7$. [2]

SECTION B

1. [8 *points*]

Solve the following equations and systems of equations. Give your answers exactly or correct to 3 significant figures.

(a) $3x + 7 = x\sqrt{2} - 2\sqrt{5}$ [2]

(b) $2x^2 - x = x + 11$ [2]

(c) $2 \times 3^x - 1 = 99$ [2]

(d) $\begin{cases} 2x - 4y = 1 \\ 4x + 5y = 12 \end{cases}$ [2]

GDC

2. [10 *points*]

Consider the following sequences:

$$a : 13,\ 21,\ 29,\ 37,\ ...$$

$$b : 10,\ 15,\ 22.5,\ 33.75,\ ...$$

(a) Decide if the sequences are arithmetic (linear), quadratic, cubic, geometric (exponential) or none of the above. Justify your answer. [2]

(b) Find the general (explicit) formula for each sequence. [4]

(c) Find the number of terms of each sequence which are smaller than 500. [4]

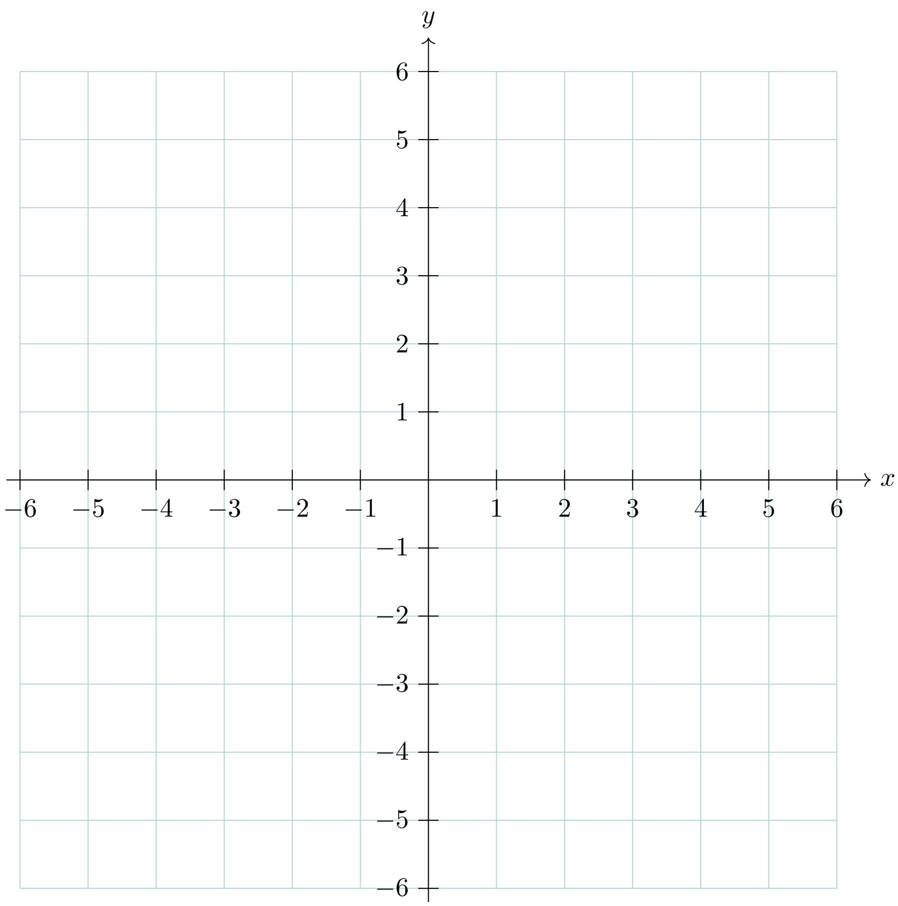
GDC

3. [6 *points*]

Consider the following system of inequalities:

$$\begin{cases} y > x \\ y \leq 3 \\ x > -1 \end{cases}$$

(a) Represent the set of points satisfying the above inequalities by shading an appropriate region on the diagram below. [3]



(b) Decide if the following points satisfy the above system of inequalities: [3]

(i) $(2, 1)$ (ii) $(0, 3)$ (iii) $(2, 4)$

4. [3 *points*]

Consider the following intervals:

$$A =]-\infty, 3] \qquad B = [-1, 4[\qquad C = [2, \infty[$$

Write down:

(a) $A - B$ [1]

(b) $B \cap C \cap \mathbb{Z}$ [1]

(c) Is it true that $2\sqrt{5} \in A \cup B$. Justify your answer. [1]

GDC

5. [9 *points*]

Consider the following quadratic sequence:

$$9,\ 9,\ 8,\ 6,\ 3,\ ...$$

(a) Write down the next (sixth) term of this sequence. [1]

(b) Find the general formula of this sequence in the form: [4]

$$u_n = an^2 + bn + c$$

(c) The last term of this sequence is equal to -181. Calculate how many terms are in this sequence. [2]

(d) Show that no term of this sequence is equal to -75. [2]

GDC